Uniform Attractors for the Non-Autonomous Benjamin-Bona-Mahony Equation

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Abstract

In this paper, we show the existence of uniform attractors for the 3D non-autonomous Benjamin-Bona-Mahony equation by establishing the uniformly asymptotical compactness.

Keywords: processes, contractive functions, uniform attractor

1 Introduction

Let $\Omega \in \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial \Omega$. We consider the non-autonomous Benjamin-Bona-Mahony (BBM) equation:

$$u_t - \Delta u_t - \nu \Delta u + \nabla \cdot \vec{F}(u) = g(t, x), \quad x \in \Omega, \quad t \in \mathbb{R}_+ = [\tau, +\infty),$$ (1)

$$u(t, x)|_{\partial \Omega} = 0,$$ (2)

$$u(\tau, x) = u_\tau(x), \quad \tau \in \mathbb{R}.$$ (3)

Here $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity vector field, $\nu > 0$ is the kinematic viscosity, $\vec{F}$ is a nonlinear vector function, $g$ is an external forcing term.
The BBM equation is a well-known model for long waves in shallow water which was introduced in Benjamin, Bona, and Mahony (1972) [2] as an improvement of the Korteweg-de Vries equation (KdV equation) for modeling long waves of small amplitude in two dimensions. Contrasting with the KdV equation, the BBM equation is unstable in its high wavenumber components. Further, while the KdV equation has an infinite number of integrals of motion, the BBM equation only has three. More results of BBM equation, we can refer [3], [4], [5], [6].

Assume that \( u_\tau \in V \), the external force \( g \in L^2_{loc}(R,H) \). For the nonlinear vector function \( \vec{F}(s) = (F_1(s), F_2(s), F_3(s)) \), \( \forall s \in R \), we denote

\[
    f_i(s) = F_i'(s), \quad \mathcal{F}_i(s) = \int_0^s F_i(r) dr,
\]

where

\[
    \vec{F}(s) = (f_1(s), f_2(s), f_3(s)), \quad \vec{F}(s) = (\mathcal{F}_1(s), \mathcal{F}_2(s), \mathcal{F}_3(s)).
\]

Assume that that \( F_i(i = 1, 2, 3) \) is a smooth function satisfying

\[
    F_i(0) = 0, \quad |F_i(s)| \leq C_1|s| + C_2|s|^2, \quad \forall s \in R,
\]

\[
    |f_i(s)| \leq C_1 + C_2|s|, \quad |\mathcal{F}_i(s)| \leq C_1|s|^2 + C_2|s|^3, \quad \forall s \in R,
\]

where \( C_1 \) and \( C_2 \) are positive constants.

In this paper, we investigate the uniform attractor for non-autonomous BBM equation, by a contractive function, the uniformly asymptotical compactness will be obtained.

## 2 Main Results

**Notations:** Throughout this paper, we set \( R_\tau = [\tau, +\infty) \), \( \tau \in R \). \( C \) stands for a generic positive constant, depending on \( \Omega \), but independent of \( t \). \( L^p(\Omega) \) \((1 \leq p \leq +\infty)\) is the generic Lebesgue space, \( H^s(\Omega) \) is the general Sobolev space. We set \( E := \{u|u \in (C^\infty(\Omega))^3, \text{div}u = 0\} \), \( H, V, W \) is the closure of the set \( E \) in the topology of \((L^2(\Omega))^3, (H^1(\Omega))^3, (H^2(\Omega))^3\) respectively. “\( \rightharpoonup \)” stands for weak convergence of sequence.

Let \( \Sigma \subseteq L^2_{loc}(R, L^2(\Omega)) \) be the hull of \( f_0 \) as a symbol space:

\[
    \Sigma = H_+(f_0) = \{f_0(t + h)|h \in R\}_{L^2_{loc}(R, L^2(\Omega))}
\]

for all \( f_0 \in L^2_{loc}(R, L^2(\Omega)) \), where \([\quad]_{L^2_{loc}(R, L^2(\Omega))}\) denotes the closure in the topology of \( L^2_{loc}(R, L^2(\Omega)) \).
Under the assumptions of the initial data, the problem (1)-(3) has a global solution \( u \in C([\tau, +\infty), V) \). \( U_f(t, \tau, u_\tau) : V \to V \) denotes the processes generated by the global solutions.

Let \( \{T(s)\} \) be the translation semigroup on \( \Sigma \), we see that the family of processes \( \{U_f(t, \tau)\} \) (\( f \in \Sigma \)) satisfies the translation identity if

\[
U_f(t+s, \tau+s) = U_{T(s)f}(t, \tau), \quad \forall \ f \in \Sigma, \ t \geq \tau, \ \tau \in R, \quad (9)
\]

\[
T(s)\Sigma = \Sigma, \quad \forall s \geq 0. \quad (10)
\]

Next, we recall a simple method to derive uniformly asymptotical compactness.

**Definition 2.1** ([7]) Let \( X \) be a Banach space and \( B \) be a bounded subset of \( X \), \( \Sigma \) be a symbol space. We call a function \( \phi(\cdot, \cdot, \cdot; \cdot, \cdot) \) defined on \( (X \times X) \times (\Sigma \times \Sigma) \) to be a contractive function on \( B \times B \) if for any sequence \( \{x_n\}_{n=1}^\infty \subset B \) and any \( \{g_n\} \subset \Sigma \), there are subsequences \( \{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty \) and \( \{g_{n_k}\}_{k=1}^\infty \subset \{g_n\}_{n=1}^\infty \) such that

\[
\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{n_k}, x_{n_l}; g_{n_k}, g_{n_l}) = 0. \quad (11)
\]

We denote the set of all contractive functions on \( B \times B \) by \( \text{Contr}(B, \Sigma) \).

**Theorem 2.2** ([7]) Let \( \{U_f(t, \tau)\} \) (\( f \in \Sigma \)) be a family of processes satisfying the translation identity on Banach space \( X \) and has a bounded uniform (w.r.t \( f \in \Sigma \)) absorbing set \( f_0 \subset X \). Moreover, assume that for any \( \varepsilon > 0 \), there exist \( T = T(B_0, \varepsilon) \) and \( \phi_T \in \text{Contr}(B_0, \Sigma) \) such that

\[
\|U_{f_1}(T, 0)x - U_{f_2}(T, 0)y\| \leq \varepsilon + \phi_T(x, y; f_1, f_2), \quad \forall \ x, y \in B_0, \ \forall \ f_1, f_2 \in \Sigma. \quad (12)
\]

Then \( \{U_f(t, \tau)\} \) (\( f \in \Sigma \)) is uniformly (w.r.t. \( f \in \Sigma \)) asymptotically compact in \( X \).

By Galerkin method and priori estimate, we easily deduce the existence of global weak solution and uniform attractor:

**Theorem 2.3** We assume that (4)-(7) hold, \( g \in L^2_{\text{loc}}(R, H) \), \( u_\tau \in V \). Then there exists a unique global weak solution of the problem (1)-(3) satisfying

\[
u \in C((\tau, T); V), \quad u_t \in L^2((\tau, T); V'). \quad (13)
\]

### 3 Discussion

**Theorem 3.1** Assume that \( g \in L^2_{\text{loc}}(R, H) \) and (4)-(7) hold, then there exists uniform attractors \( A_f(t) \) in \( V \) for the non-autonomous system (1)-(3).
Proof. We shall prove the result by two steps as follows, the first one is to get the existence of an absorbing ball, the second is to prove the asymptotical compactness by means of contractive functions.

From the property of solutions, we easily obtain

Lemma 3.2 The set class \( \{ U_g(t, \tau, u_\tau) \} \) \( (\tau \leq t) \) is a process in \( V \) for all \( \tau \leq t \). Moreover, the mapping \( U_g(t, \tau, u_\tau) : V \to V \) is continuous.

Lemma 3.3 Assume that (4)-(7) hold, then there exists a global uniform (w.r.t. \( g \in \Sigma \)) absorbing set \( B_0 \) of the process \( \{ U_g(t, \tau, u_\tau) \} \).

Proof. For all \( u \in V \), multiplying both sides of (1) with \( u \) and noticing that

\[
\int_\Omega (\nabla \cdot F(u)) u dx = -\int_\Omega F(u) \nabla u dx = -\int_\Omega \nabla \cdot \bar{F}(u) dx = 0,
\]

we derive

\[
\frac{d}{dt}(\| u(t) \|^2 + \| \nabla u(t) \|^2) + 2\nu\| \nabla u(t) \|^2 = 2(g(t), u(t)) \\
\leq 2\nu\| \nabla u(t) \|^2 + \frac{2}{\nu\lambda} \| g(t) \|^2.
\]

Consequently, for all \( \tau \in R \)

\[
\| u(t) \|^2 + \| \nabla u(t) \|^2 \leq (\| u_\tau \|^2 + \| \nabla u(\tau) \|^2) + \frac{2}{\nu\lambda} \int_\tau^t \| g(\xi) \|^2 d\xi.
\]

Setting \( r^2 = \| u_\tau \|^2 + \| \nabla u_\tau \|^2 \), we easily get

\[
\| U_g(t, \tau, u_\tau) \|^2_V \leq r^2 + \frac{2}{\nu\lambda} \int_\tau^t \| g(\xi) \|^2 d\xi,
\]

for all \( u_\tau \in V, \ t \geq \tau \).

Setting

\[
r^2 \leq \frac{2}{\nu\lambda} \int_{-\infty}^t \| g(\xi) \|^2 d\xi,
\]

then we denote \( R \) the nonnegative number given by

\[
R^2 = \frac{2}{\nu\lambda} \int_{-\infty}^t \| g(\xi) \|^2 d\xi,
\]

and consider the family of closed balls \( B_0 \) in \( V \) defined by

\[
B_0 = \{ v \in V \| v \|^2_V \leq 2R \}.
\]

It is straightforward to check that \( B_0 \) is a uniform absorbing ball for the process \( \{ U_g(t, \tau, u_\tau) \} \).
Lemma 3.4 We assume that \( \{u^n_\tau\} \) is a sequence in \( V \) and weakly converges to \( u_\tau \in V \). Then

\[
U_g(t, \tau, u^n_\tau) \rightarrow U_g(t, \tau, u_\tau) \text{ weakly in } H, \quad \forall \ t \geq \tau, \tag{20}
\]

\[
U_g(\cdot, \tau, u^n_\tau) \rightarrow U_g(\cdot, \tau, u_\tau) \text{ weakly in } L^2(\tau, T; V), \quad \forall \ t \geq \tau. \tag{21}
\]

Proof. From the property of the solution, i.e., the solution is boundedness in appropriate topology, we easily conclude the result.

Lemma 3.5 Under the conditions of (4)-(7), the process \( \{U_g(t, \tau, u_\tau)\} \) generated by the global solutions is uniform asymptotically compact, i.e., the sequence \( U_g(t, \tau_n, u^n_\tau) \) has a subsequence which is strongly convergent in \( V \).

Proof. For any initial data \( u_i^\tau \in B_0 \ (i = 1, 2) \), let \( u^i(t, x) \) be the corresponding solutions to the symbols \( g^i \) with \( u_i^\tau \), that is, \( u^i(t) \) is the solution of the equation:

\[
u t - \Delta u + \nu \Delta u + \nabla \cdot \overrightarrow{F}(u) = g^i(t, x), \tag{22}
\]

\[
u(t, x)|_{\partial \Omega} = 0, \tag{23}
\]

\[
u(\tau, x) = u_i^\tau(x), \quad \tau \in \mathbb{R}. \tag{24}
\]

Denote

\[w(t) = u_1(t) - u_2(t), \tag{25}\]

then \( w(t) \) satisfies

\[
u_t - \Delta w - \nu \Delta w + \nabla \cdot \left[ \overrightarrow{F}(u_1(t)) - \overrightarrow{F}(u_2(t)) \right] = g^1(t, x) - g^2(t, x), \tag{26}
\]

\[
u(t, x)|_{\partial \Omega} = 0, \tag{27}
\]

\[
u(\tau, x) = u_1^\tau(x) - u_2^\tau(x), \quad \tau \in \mathbb{R}. \tag{28}
\]

Setting

\[E_w(t) = \frac{1}{2} \int_\Omega |w(t)|^2 dx + \frac{1}{2} \int_\Omega |\nabla w(t)|^2 dx. \tag{29}\]

Multiplying (26) by \( w \) and integrating over \([s, T] \times \Omega\), we deduce

\[
u_w(T) - E_w(s) + \nu \int_s^T \int_\Omega |\nabla w(h)| dx dh
\]

\[+ \int_s^T \int_\Omega \left( \overrightarrow{F}(u_1(h)) - \overrightarrow{F}(u_2(h)) \right) w(h) dx dh
\]

\[= \int_s^T \int_\Omega \left( g_1(h) - g_2(h) \right) w(h) dx dh, \tag{30}\]
where $\tau \leq s \leq T$. Then we have

$$
\nu \int_{\tau}^{T} \int_{\Omega} |\nabla w(h)| dx dh \leq E_w(\tau) - \int_{\tau}^{T} \int_{\Omega} (F(u_1(h)) - F(u_2(h))) w(h) dx dh \\
+ \int_{\tau}^{T} \int_{\Omega} (g_1(h) - g_2(h)) w(h) dx dh.
$$

Hence,

$$
\int_{\tau}^{T} E_w(s) ds = \int_{\tau}^{T} \left( \frac{1}{2} \int_{\Omega} |w(s)|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla w(s)|^2 dx \right) ds \\
\leq C \int_{\tau}^{T} \int_{\Omega} |\nabla w(s)|^2 dx ds \\
\leq C \left[ E_w(\tau) - \int_{\tau}^{T} \int_{\Omega} (F(u_1(s)) - F(u_2(s))) w(s) dx ds \\
+ \int_{\tau}^{T} \int_{\Omega} (g_1(s) - g_2(s)) w(s) dx ds \right].
$$

Integrating (30) over $[\tau, T]$ with respect to $s$, we get

$$
TE_w(T) + \nu \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} |\nabla w(h)| dx dh ds dh \\
\leq \int_{\tau}^{T} E_w(s) ds - \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (F(u_1(h)) - F(u_2(h))) w(h) ds dh ds \\
+ \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (g_1(h) - g_2(h)) w(h) dx dh ds dh \\
\leq C \left[ E_w(\tau) - \int_{\tau}^{T} \int_{\Omega} (F(u_1(s)) - F(u_2(s))) w(s) dx ds \\
+ \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (g_1(s) - g_2(s)) w(s) dx ds \right] \\
- \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (F(u_1(h)) - F(u_2(h))) w(h) ds dh ds \\
+ \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (g_1(h) - g_2(h)) w(h) dx dh ds. \quad (33)
$$

If we set

$$
C_0 = CE_w(\tau),
$$

$$
\phi(u_0, u_0^2; g^1(t), g^2(t)) = C \left[ - \int_{\tau}^{T} \int_{\Omega} (F(u_1(s)) - F(u_2(s))) w(s) dx ds \\
+ \int_{\tau}^{T} \int_{\Omega} (g_1(s) - g_2(s)) w(s) dx ds \right] \\
- \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (F(u_1(h)) - F(u_2(h))) w(h) ds dh ds \\
+ \int_{\tau}^{T} \int_{\tau}^{s} \int_{\Omega} (g_1(h) - g_2(h)) w(h) dx dh ds, \quad (34)
$$
then we have
\[ E_w(T) \leq \frac{C_0}{T} + \frac{1}{T} \phi(u_0^1, u_0^2, g^1(t), g^2(t)). \]  
(35)

Since the family of process has a uniformly bounded absorbing set, we choose \( T \) large enough such that
\[ \frac{C_0}{T} \leq \varepsilon, \]  
(36)
i.e., \( T \geq \frac{C_0}{\varepsilon} \).

Let \( u^n, \ u^l \) be the solutions with respect to the initial data \( u_0^n, u_0^l \) and symbols \( g^n(t), g^l(t) \in \Sigma, \ m, \ n = 1, 2, \ldots \) respectively. Then from Theorem 2.3, we can derive
\[
\lim_{n \to \infty} \lim_{l \to \infty} \int_T^0 \int_\Omega (F(u^n(s)) - F(u^l(s)))(u^n(s) - u^l(s))dxds = 0,
\]
\[
\lim_{n \to \infty} \lim_{l \to \infty} \int_T^0 \int_\Omega (g^n(s) - g^l(s))(u^n(s) - u^l(s))dxds = 0,
\]
\[
\lim_{n \to \infty} \lim_{l \to \infty} \int_T^0 \int_\Omega (F(u^n(h)) - F(u^l(h)))(u^n(s) - u^l(s))dshds = 0,
\]
\[
\lim_{n \to \infty} \lim_{l \to \infty} \int_T^0 \int_\Omega (g^n(h) - g^l(h))(u^n(s) - u^l(s))dxdhds = 0. \]  
(37)
Hence \( \phi(u_0^1, u_0^2; g^1(t), g^2(t)) \in \text{Contr}(B_0, \Sigma) \) for the above \( T \). By Theorem 2.2, the conclusion holds.

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**References**


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