On the First Hitting Times to Boundary of the Reflected O-U Process with Two-Sided Barriers

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Abstract. In this paper, we give the Laplace transform of the first hitting times to boundary and obtain its expectation on the reflected Ornstein-Uhlenbeck process with two-sided barriers. Finally, a comparison theorem is derived.

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1 Introduction

Recently the reflected Ornstein-Uhlenbeck (ROU) process has extensively been investigated by many authors. ROU has many applications in different area such as queueing systems, credit risk modeling. In queueing systems, ROU with one barrier can be a limit process of a special queueing systems. Ward & Glynn (2003) discussed a queueing system with reneging and constructed a ROU process with one barrier via an appropriate Markovian approximation procedure. In their successive article properties of the reflected process was discussed. In passing, Bo (2006) consider a fluid model with finite buffer capacity. This leads to the ROU process with two-sided barriers. Zhang (2009, 2010) discuss properties of ROU with two barriers, the Laplace transform of the first hitting times and stationary distribution was obtained separately. Furthermore, ROU with two barriers was also used to model the price dynamics of the regulated goods or services, Bo (2011) investigated Conditional default probability in a regulated market.

The so-called ROU with two barriers is as follows. Let \( Z = \{Z_t, t \geq 0\} \) be a one-dimensional reflected Ornstein-Uhlenbeck process with barriers 0 and \( b \) (\( b > 0 \) is a constant), which is defined by

\[
\begin{aligned}
dZ_t &= (\mu - \alpha Z_t) dt + \sigma dB_t + dL_t - dU_t, \\
Z_0 &= x \in [0, b],
\end{aligned}
\]  

(1.1)

where \( B = \{B_t, t \geq 0\} \) is an one-dimensional standard Brownian motion, and \( \mu \in \mathbb{R}, \alpha, \sigma \in \mathbb{R}^+ \). Here \( L = \{L_t, t \geq 0\} \) and \( U = \{U_t, t \geq 0\} \) are the regulators of point 0 and \( b \), respectively. Further, the processes \( L \) and \( U \) are uniquely determined by the following properties (see Harrison, M (1986)),

- Both \( t \to L_t \) and \( t \to U_t \) are continuous nondecreasing processes with \( L_0 = U_0 = 0 \),

- \( L \) and \( U \) increase only when \( Z = 0 \) and \( Z = b \), respectively, i.e.,

\[
\int_0^t \mathbb{1}_{(Z_s=0)} dL_s = L_t \quad \text{and} \quad \int_0^t \mathbb{1}_{(Z_s=b)} dU_s = U_t, \quad \text{for} \ t \geq 0.
\]

Lions & Sznitman (1984) guaranteed the existence and uniqueness of the solutions of (1.1). The first passage times or hitting times are important quantities in applications of ROU. Bo (2006) and Zhang (2010) obtained the Laplace transform of the first passage times of the O-U process with two-sided barriers. In this paper, we focus on the first hitting times to boundary.

The paper is organized as follows. In Section 2, we investigate the first hitting times and obtain the Laplace transform of the first hitting times to boundary. Section 3 provide the result on the expectation of of the first hitting times. In the final section, a comparison theorem is obtained.
2 Laplace transform of the first hitting times

In this section we consider the equation (1.1) with assumptions \(Z_0 = x \in [0, b]\). Let \(y \in [0, b]\), define the first hitting time by
\[
T(y) := \inf\{t \geq 0 : Z_t = y\},
\]
with the usual convention \(\inf \emptyset = \infty\). On the other hand, suppose \(\lambda > 0\) and \(f \in C^2([0, b])\), define a linear operator
\[
Af(x) := \frac{\sigma^2}{2} f''(x) + (\mu - \alpha x) f'(x) - \lambda f(x), \quad \text{for } x \in [0, b].
\]
For \(0 \leq x \leq b\), we are going to give the expression of the Laplace transform of \(T^* = T(0) \wedge T(b)\).

**Theorem 2.1** Suppose that \(f(\lambda)\) is the solution of the following equation
\[
Af(y) = 0, \quad y \in [0, b], \quad \text{and } f(0) = f(b),
\]
then,
\[
\mathbb{E}_x [e^{-\lambda T}] = \frac{f(\lambda)(x)}{f(\lambda)(0)}. \tag{2.2}
\]

**Proof.** Applying Itô formula for \(h(t, x) := \exp(-\lambda t) f(x)\) with \(f \in C^2_b([0, b])\), we have
\[
\begin{align*}
  h(t, Z_t) &= h(0, Z_0) + \int_0^t \frac{\partial h}{\partial s}(s, Z_s)ds + \int_0^t \frac{\partial h}{\partial x}(s, Z_s)dZ_s \\
  &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 h}{\partial x^2}(s, Z_s)d\langle Z, Z \rangle_s \\
  &= f(Z_0) + \int_0^t e^{-\lambda s} \left[ -\lambda f(Z_s) - \alpha Z_s f'(Z_s) + \frac{\sigma^2}{2} f''(Z_s) \right] ds \\
  &\quad + \int_0^t e^{-\lambda s} f'(Z_s)dL_s - \int_0^t e^{-\lambda s} f'(Z_s)dU_s + \sigma \int_0^t e^{-\lambda s} f'(Z_s)dB_s \\
  &= f(Z_0) + \int_0^t e^{-\lambda s} A(\lambda) f(Z_s)ds + f'(0) \int_0^t e^{-\lambda s}dL_s \\
  &\quad - f'(b) \int_0^t e^{-\lambda s}dU_s + \sigma \int_0^t e^{-\lambda s} f'(Z_s)dB_s, \tag{2.3}
\end{align*}
\]

since \(L\) and \(U\) are finite variation (FV) processes. Let \(T < \infty\) be a stopping time and \(x \in [0, b]\). It follows from martingale optional theorem, that
\[
\begin{align*}
  \mathbb{E}_x [e^{-\lambda T} f(Z_T)] &= f(x) + \mathbb{E}_x \left[ \int_0^T e^{-\lambda s} A(\lambda) f(Z_s)ds \right] \\
  &\quad + f'(0) \mathbb{E}_x \left[ \int_0^T e^{-\lambda s}dL_s \right] - f'(b) \mathbb{E}_x \left[ \int_0^T e^{-\lambda s}dU_s \right] .
\end{align*}
\]
In particular, take $T^\ast = T^\ast$ for $y \in [0, b]$, and note that

$$
\mathbb{E}_x \left[ \int_0^{T^\ast} e^{-\lambda s} dU_s \right] = 0,
$$

and

$$
\mathbb{E}_x \left[ \int_0^{T^\ast} e^{-\lambda s} dL_s \right] = 0.
$$

Then,

$$
\mathbb{E}_x \left[ e^{-\lambda T^\ast} f(Z_{T^\ast}) \right] = f(x) + \mathbb{E}_x \left[ \int_0^{T^\ast} e^{-\lambda s} Af(Z_s) ds \right]
$$

(2.4)

Replace $f$ by $f(\lambda)$ in (2.4), we immediately get (2.2) by $Z_{T^\ast} = 0$ or $b$, $f(\lambda)(0) = f(\lambda)(b)$. Thus the proof of the theorem is completed. □

**Remark 2.1** For the Laplace transform of $T^\ast$ can be given by $f$ which satisfies $Af(x) = 0$ and $f(0) = f(b)$, we must find the solution of second order equation $Af(x) = 0$ with initial condition $f(0) = f(b)$.

Next, we are concerned with the second order equation $Af(x) = 0$ with initial condition $f(0) = f(b)$.

Notice that

$$
Af(x) := \frac{\sigma^2}{2} f''(x) + (\mu - \alpha x) f'(x) - \lambda f(x),
$$

so $Af(x) = 0$ with initial condition $f(0) = f(b)$ equals to

$$
\frac{\sigma^2}{2} f''(x) + (\mu - \alpha x) f'(x) - \lambda f(x) = 0, f(0) = f(b)
$$

(2.5)

Using the variable transformation $z = \alpha x - \mu$ and the notation $g(z) = f\left(\frac{z + \mu}{\alpha}\right)$, (2.5) can be reduced to the following equation

$$
g''(z) - azg'(z) - bg(z) = 0, g(-\mu) = g(\alpha b - \mu),
$$

(2.6)

where

$$
a = \frac{2}{\alpha \sigma^2}, b = \frac{2\lambda}{\alpha^2 \sigma^2}.
$$

According to the theory of second order linear ODE, the general solution of (2.6) can be given by the following lemma.
Lemma 2.1 \( g(z) = C \left\{ A \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} (2ai+b)}{(2n)!} z^{2n} + \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} (2ai+a+b)}{(2n+1)!} z^{2n+1} \right\} \) is the general solution of \( g''(z) - azg'(z) - bg(z) = 0 \), \( g(-\mu) = g(\alpha b - \mu) \), where

\[
A = \sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-2} (2ai+a+b)}{(2n)!} \left( \mu^{2n+1} - (\mu - \alpha b)^{2n+1} \right) \text{ and } C \text{ is the constant.}
\]

Proof. \( g''(z) - azg'(z) - bg(z) = 0 \) is a second order linear ODE, so we must find two linearly independent solutions and the general solution will be a linear combination of them. We exploit the method of Taylor’s series to obtain two linearly independent solutions. Indeed two linearly independent solutions can be expressed by Taylor’s series,

\[
g_1(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} (2ai+b)}{(2n)!} z^{2n}, \quad g_2(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{n-1} (2ai+a+b)}{(2n+1)!} z^{2n+1}.
\]

So the linear combination of \( g_1 \) and \( g_2 \) is the general solution of \( g''(z) - azg'(z) - bg(z) = 0 \). Noting that \( g(-\mu) = g(\alpha b - \mu) \), the general solution can be obtained easily.

\[\square\]

Theorem 2.2

\[ f^{(\lambda)}(x) = C \left\{ A g_1(\alpha x - \mu) + g_2(\alpha x - \mu) \right\} \]

is the general solution of

\[
\frac{\sigma^2}{2} f''(x) + (\mu - \alpha x) f'(x) - \lambda f(x) = 0, \quad f(0) = f(b), \quad (2.7)
\]

where \( A = \sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-2} (2ai+a+b)}{(2n)!} \left( \mu^{2n+1} - (\mu - \alpha b)^{2n+1} \right) \) and \( C \) is the constant.

Proof. Noting that variable transformation \( z = \alpha x - \mu \) and the notation \( g(z) = f_1(\frac{z}{\alpha} + \frac{\mu}{\alpha}) \), the accuracy of this theorem can be proved. \[\square\]

3 Expectation of the first hitting time

In this section we give the analytic expression of the expectation of the first hitting times. In convenience, define a operator

\[ Af(x) := \frac{\sigma^2}{2} f''(x) + (\mu - \alpha x) f'(x), \quad \text{for } x \in [0, b]. \]

With it, we have the following theorem.
Theorem 3.1 For $x \in [0, b]$ ,

$$E_x(T^*) = \sqrt{\frac{4\pi}{\alpha \sigma^2}} \int_0^x \exp\left(\frac{(\alpha v - \mu)^2}{\alpha \sigma^2}\right) P\left(0 \leq N\left(\frac{\mu}{\alpha}, \frac{\sigma^2}{2\alpha}\right) \leq v\right) dv \quad (3.1)$$

$$- D \sqrt{\frac{4\pi}{\alpha \sigma^2}} \int_0^x \exp\left(\frac{(\alpha v - \mu)^2}{\alpha \sigma^2}\right) dv \quad (3.2)$$

where

$$D = \frac{\int_0^b \exp\left(\frac{(\alpha v - \mu)^2}{\alpha \sigma^2}\right) P\left(0 \leq N\left(\frac{\mu}{\alpha}, \frac{\sigma^2}{2\alpha}\right) \leq v\right) dv}{\int_0^b \exp\left(\frac{(\alpha v - \mu)^2}{\alpha \sigma^2}\right) dv} \quad (3.3)$$

and $N\left(\frac{\mu}{\alpha}, \frac{\sigma^2}{2\alpha}\right)$ denote the normal random variable with mean $\frac{\mu}{\alpha}$ and variance $\frac{\sigma^2}{2\alpha}$.

**Proof.** Applying Itô formula for twice differentiable function $h(x)$ and recalling the notation of the operator $A$, we have

$$h(Z_t) = h(Z_0) + \int_0^t \frac{\partial h}{\partial x}(Z_s) dZ_s + \frac{1}{2} \int_0^t \frac{\partial^2 h}{\partial x^2}(Z_s) d\langle Z, Z \rangle_s$$

$$= h(Z_0) + \int_0^t \left[ (\mu - \alpha z_s) h'(Z_s) + \frac{\sigma^2}{2} h''(Z_s) \right] ds$$

$$+ \int_0^t h'(Z_s) dL_s - \frac{1}{2} \int_0^t h''(Z_s) dB_s$$

$$= h(Z_0) + \int_0^t Ah(Z_s) ds + h'(0)L_s - h'(b)U_s + \sigma \int_0^t h'(Z_s) dB_s$$

The last equation holds, for $L$ and $U$ increase only when $Z = 0$ and $Z = b$ respectively. Let $T < \infty$ be a stopping time and $Z_0 = x \in [0, b]$. It follows from martingale optional theorem, that

$$E_x(h(Z_T)) = h(x) + E_x\left[ \int_0^T Ah(Z_s) ds \right] + h'(0)E_x(L_T) - h'(b)E_x(U_T).$$

In particular, take $T = T^*$ for $y \in [0, b]$, and note that

$$E_x(U_{T^*}) = 0, E_x(L_{T^*}) = 0.$$

Then,

$$E_x(h(Z_{T^*})) = h(x) + E_x\left[ \int_0^{T^*} Ah(Z_s) ds \right] \quad (3.4)$$
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Denoting by

\[ h^*(x) = h_1(x) - Dh_2(x), \]

where

\[ h_1(x) = \sqrt{\frac{4\pi}{\alpha\sigma^2}} \int_0^x \exp \left( \frac{(\alpha v - \mu)^2}{\alpha^2} \right) P \left( 0 \leq N \left( \frac{\mu}{\alpha}, \frac{\sigma^2}{2\alpha} \right) \leq v \right) dv \]

and

\[ h_2(x) = \sqrt{\frac{4\pi}{\alpha\sigma^2}} \int_0^x \exp \left( \frac{(\alpha v - \mu)^2}{\alpha^2} \right) dv, \]

we immediately obtain \( Ah^*(x) = -1, h^*(0) = h^*(b) = 0 \). Replacing \( h \) by \( h^* \) in (3.4) and noting \( Z_{T^*} = 0 \) or \( b \), Theorem 3.1 can be proved easily. Thus the proof of the theorem is completed.

\[ \square \]

4 Comparison of the first hitting times

This section mainly focus on a comparison theorem, namely the relation between \( T(0) \) and \( T(b) \).

**Theorem 4.1** For \( x \in [0, b] \), we have

\[ P(T(0) < T(b)) = \frac{\int_x^b \exp \left( \frac{(\alpha v - \mu)^2}{\alpha^2} \right) dv}{\int_0^b \exp \left( \frac{(\alpha v - \mu)^2}{\alpha^2} \right) dv} \]  

(4.1)

**Proof.** Denoting by \( u(x) = \int_0^x \exp \left( \frac{(\alpha v - \mu)^2}{\alpha^2} \right) dv \), and applying Itô formula for \( u(x) \), we have

\[
\begin{align*}
    u(Z_t) &= u(Z_0) + \int_0^t \frac{\partial u}{\partial x}(Z_s)dZ_s + \frac{1}{2} \int_0^t \frac{\partial^2 u}{\partial x^2}(Z_s)d\langle Z, Z \rangle_s \\
    &\quad + \int_0^t u'(Z_s)dL_s - \int_0^t u'(Z_s)dU_s + \sigma \int_0^t u'(Z_s)dB_s \\
    &= u(Z_0) + \int_0^t \left[ (\mu - \alpha z_s)u'(Z_s) + \frac{\sigma^2}{2} u''(Z_s) \right] ds \\
    &\quad + \int_0^t u'(Z_s)dL_s - \int_0^t u'(Z_s)dU_s + \sigma \int_0^t u'(Z_s)dB_s.
\end{align*}
\]

The last equation holds, for \( L \) and \( U \) increase only when \( Z = 0 \) and \( Z = b \) respectively. For the stopping time \( T^* \) and \( Z_0 = x \in [0, b] \). It follows from martingale optional theorem, that

\[
E_x (u(Z_{T^*})) = u(x) + E_x \left[ \int_0^{T^*} Au(Z_s)ds \right] + u'(0)E_x (L_{T^*}) - u'(b)E_x (U_{T^*}).
\]
In particular, note that
\[ E_x(u(Z_{T^*})) = u(0) \times P(T(0) < T(b)) + u(b) \times P(T(0) > T(b)) \]
and
\[ Au(x) = 0, u(0) = 0, E_x(L_{T^*}) = 0, E_x(U_{T^*}) = 0, \]
we immediately obtain
\[ P(T(0) < T(b)) = \frac{\int_x^b \exp \left( \frac{(\alpha v - \mu)^2}{2\sigma^2} \right) dv}{\int_0^b \exp \left( \frac{(\alpha v - \mu)^2}{2\sigma^2} \right) dv}. \]
Thus the proof of the theorem is completed.

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