The Itô Exponential on Lie Groups

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Abstract

Let $G$ be a Lie Group with a complete, left invariant connection $\nabla^G$. Denote by $\mathfrak{g}$ the Lie algebra of $G$ which is equipped with a complete connection $\nabla^\mathfrak{g}$. Our main goal is to introduce the concept of the Itô exponential and the Itô logarithm. As a result, we characterize the martingales in $G$ with respect to the left invariant connection $\nabla^G$. Also, assuming that connection function $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ associated to $\nabla^G$ satisfies $\alpha(M, M) = 0$ for all $M \in \mathfrak{g}$ we obtain a stochastic Campbell-Hausdorff formula. Further, from this stochastic Campbell-Hausdorff formula we present a way to construct martingales in Lie group. In consequence, we show that a product of harmonic maps with value in $G$ is a harmonic map. To end, we apply this study in some matrix Lie groups.

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1 Introduction

M. Hakim-Dowek and D. Lépingle introduced the exponential stochastic in Lie Groups by first time in [10]. Their idea can be interpreted in the following way. Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Given a semimartingale $M$
in \( g \), the stochastic exponential \( X = e(M) \) is the solution, in the Stratonovich sense, of the stochastic differential equation

\[
\delta X = L_{X^*} \delta M,
\]

\( X_0 = e \).

Furthermore, they showed a existance of the inverse of the stochastic exponential, which is known as the stochastic logarithm. In this paper [10] was developed a serial of result about the stochastic exponential and the stochastic logarithm.

The concept of the stochastic exponential and the stochastic logarithm in Lie group has been studied and applied in some situations. For example, M. Arnaudon developed studies of the stochastic exponential in Lie groups in the case that \( G \) has a left invariant connection [1]. He also used the stochastic exponential to study the martingales and Brownian motions in homogeneous space [2]. The characterization of the semimartingales, martingales and Brownian motions in a principal fiber bundle, due to M. Arnaudon and S. Paycha [3], is obtained with the stochastic exponential. In the case that a Lie Group \( G \) has a bi-invariant metric, P. Catuogno and P. Ruffino in [4] used the stochastic exponential and the stochastic logarithm to show that the product of harmonic maps with values in \( G \) is a harmonic map.

The developed of the stochastic exponential has been done without to takes in account some geometry of the Lie groups. Despite of the above studies have worked with some types of connections. In a nutshell, this fact occurs because the integral of Stratonovich does not have intrinsically the geometry of the smooth manifolds. Unlike, the integral of Itô on a smooth manifold intrinsically has the information of the geometry of the smooth manifold. Considering this fact we introduce a stochastic exponential and a stochastic logarithm in the Itô sense.

Let \( \nabla^G \) be a complete, left invariant connection on \( G \) and \( \nabla^g \) a complete connection on \( g \). The Itô exponential and the Itô logarithm are the solutions of the following stochastic differential equations, respectively,

\[
d^G X = L_{X^*} d^g M, \quad X_0 = e
\]

and

\[
d^g N = L_{Y^{-1}*} d^G Y, \quad Y_0 = 0.
\]

We denote this solution by \( e^{Gg}(M) \) and \( L^{Gg}(Y) \), respectively.

Our first work is to show that this equations have unique solutions that do not explode in a finite time, since \( \nabla^G \) and \( \nabla^g \) are completes. Also, we show
that the operators $e^{G\theta}$ and $L^{G\theta}$ are inverses. As a result, we get that every $\nabla^G$-martingales is given by $e^{G\theta}(M)$ for a $\nabla^\theta$-martingale $M$ in $\mathfrak{g}$.

An observable fact is that the Itô exponential and Itô logarithm are dependents of the connections $\nabla^G$ and $\nabla^\theta$, however it is not necessary a kind of the correspondence between these connections.

Other work is to construct a stochastic Campbell-Hausdorf formula. Given a left invariant connection $\nabla^G$ there exists a unique bilinear form $\alpha$ on $\mathfrak{g}$ associated to $\nabla^G$ (see for example [11]). If we suppose that $\alpha(M,M) = 0$ for all $M \in \mathfrak{g}$, then $\nabla^G$ is complete. Under this hypothesis, with property of null quadratic variation property, which is defined in 4.1, we show a stochastic Campbell-Hausdorff formula. This formula help us to create a way to construct martingales in the Lie group with respect to $\nabla^G$. A little bit, from this result we can show that a product of harmonic maps with values in $G$ is a harmonic map.

To end, we show a relation between Itô logarithm and stochastic logarithm. This relation will give us the geometry of Lie group in therms of the Itô logarithm. We apply this result in the study of the martingales in some matrix Lie group equipped with a class of the left invariant metric, namely, the Euclidian motion group $SE(3)$ and the three-dimensional non compact Lie Groups $SE(2)$, $E(1,1)$, $N^3$ and $SL(2,\mathbb{R})$.

2 Preliminaries

In this work we use freely the concepts and notations of P. Protter [15], E. Hsu [12], P. Meyer [14], M. Emery [7] and [9], and S. Kobayashi and N. Nomizu [13]. We suggest the reading of [5] for a complete survey about the objects of this section. From now on the adjective smooth means $C^\infty$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for example [7]). Our basic assumption is that every stochastic process is continuous.

Let $M$ be a smooth manifold and $X_t$ a continuous stochastic process with values in $M$. We call $X_t$ a semimartingale if, for all $f$ smooth function, $f(X_t)$ is a real semimartingale.

Let $M$ be a smooth manifold endowed with a symmetric connection $\nabla^M$. From now on we make the assumption: all connections will be symmetrics. Let $X$ be a semimartingale in $M$ and $\theta$ a 1-form on $M$ defined along $X$. Let $(U, x_1, \ldots, x_n)$ be a local coordinate system on $M$. We define the Stratonovich and the Itô integrals, respectively, of $\theta$ along $X$, locally, by

$$\int_0^t \theta \delta X_s = \int_0^t \theta_i(X_s) dX^i_s + \frac{1}{2} \int_0^t \frac{\partial \theta_i}{\partial x^j}(X_s) d[X^i_s, X^j_s],$$
and
\[ \int_0^t \theta^M X_s = \int_0^t \theta^i(X_s) dX^i_s + \frac{1}{2} \int_0^t \Gamma^i_{jk}(X_s) \theta^i(X_s) d[X^j, X^k]_s, \tag{1} \]
where \( \theta = \theta^i dx^i \) with \( \theta^i \) smooth functions and \( \Gamma^i_{jk} \) are the Christoffel symbols of the connection \( \nabla^M \). Let \( b \in T^{(2,0)}M \) be defined along \( X \). We define the quadratic integral on \( M \) along \( X \), locally, by
\[ \int_0^t b^i(X, dX)_s = \int_0^t b_{ij}(X_s) d[X^i, X^j]_s, \]
where \( b = b_{ij} dx^i \otimes dx^j \) with \( b_{ij} \) smooth functions.

A direct consequence of the definitions above is the Stratonovich-Itô formula of conversion given by
\[ \int_0^t \theta \delta X_s = \int_0^t \theta^M X_s + \frac{1}{2} \int_0^t \nabla^M \theta^i (dX^i, dX)_s. \tag{2} \]

A semimartingale \( X \) with values in \( M \) is called a \( \nabla^M \)-martingale if \( \int \theta^M X \) is a real local martingale for all \( \theta \in \Gamma(TM^*) \).

Let \( M \) be a Riemannian manifold with a metric \( g \). A semimartingale \( B \) in \( M \) is said a \( g \)-Brownian motion if \( B \) is a \( \nabla^g \)-martingale, being \( \nabla^g \) the Levi-Civita connection of \( g \), and for any section \( b \) of \( T^{(2,0)}M \) we have
\[ \int_0^t b dB, dB)_s = \int_0^t \text{tr} b(B_s) ds. \tag{3} \]

Let \( M \) and \( N \) be smooth manifolds with connections \( \nabla^M \) and \( \nabla^N \), respectively, \( \theta \) a section of \( TN^* \), \( b \) a section of \( T^{(2,0)}N \) and \( F : M \to N \) a smooth map. The geometric Itô formula is given by
\[ \int_0^t \theta F(X_s) = \int_0^t F^* \theta^M X_s + \frac{1}{2} \int_0^t \beta^*_F \theta^i (dX^i, dX)_s, \tag{4} \]
where \( \beta^*_F \) is the second fundamental form of \( F \) with respect to \( \nabla^M \) and \( \nabla^N \).

## 3 The Itô exponential and Itô logarithm

Let \( G \) be a Lie Group and \( \mathfrak{g} \) its Lie algebra. Let us denote by \( L_g \) the left translation on \( G \). From this we can construct the following family of linear applications on \( \mathfrak{g}^* \otimes TG \): since \( \mathfrak{g} \) is isomorphic to \( T_e \mathfrak{g} \), we consider that the left translation is a linear application \( L_{g*}(e) : \mathfrak{g} \to TG \) for every \( g \in G \).

We observe that this family of applications is smooth in the following sense.
Taking $E \in \mathfrak{g}$ we obtain a smooth left invariant vector field $X \in TG$ such that $L_{g^*}(e)(E) = X_g$. Therefore $L_{g^*}(e)$ is a smooth family from $\mathfrak{g} \times G$ into $TG$ (see for instant Definition 6.34 in [7]).

We endow $G$ with a left-invariant connection $\nabla^G$ and $\mathfrak{g}$ with a connection $\nabla^\mathfrak{g}$. Let $X$ be a semimartingale in $G$ and $M$ a semimartingale in $\mathfrak{g}$. One says that $X$ is a solution to the stochastic differential equation
\[
d^G X_t = L_{X_t^*}(e) d^\mathfrak{g} M_t, \tag{5}
\]
if, for every 1-form $\theta$ on $G$, the real semimartingales $\int_0^t \theta d^G X_s$ and $\int_0^t L_{X_t^*}(e) \theta d^\mathfrak{g} M_s$ are equal.

Firstly, one may observe that the solution of the stochastic differential equation above is invariant because of the left invariance of the connection $\nabla^G$.

**Proposition 3.1** Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Assume that $\nabla^G$ is a left-invariant connection on $G$ and $\nabla^\mathfrak{g}$ is a connection on $\mathfrak{g}$. Suppose that $Y_t$ is a solution of (5). If $\xi$ is a random variable with values in $G$, then $X_t = \xi Y_t$ is also a solution of (5).

**Proof:** We begin denoting the product on Lie group $G$ by $m$. Let $\theta$ be a 1-form on $G$. As a function to $m$, the integral of Itô along $X_t$ is writing by
\[
\int_0^t \theta d^G X_s = \int_0^t \theta d^G \xi Y_s = \int_0^t \theta d^G m(\xi, Y_s).
\]

The geometric Itô formula (4) get
\[
\int_0^t \theta d^G X_s = \int_0^t m^* \theta d^{G \times G}(\xi, Y_s) + \frac{1}{2} \int_0^t \beta^* m \theta(d(\xi, Y_s), d(\xi, Y_s)).
\]

From Proposition 3.15 in [9] we see that
\[
\int_0^t \theta d^G X_s = \int_0^t (R_{Y_s^*} \theta) d^G \xi + \int_0^t (L_{\xi^*} \theta) d^G Y_s + \frac{1}{2} \int_0^t \beta^* m \theta(d(\xi, Y_s), d(\xi, Y_s)).
\]

We see that $\xi$ is a constant process, consequently,
\[
\int_0^t \theta d^G X_s = \int_0^t (L_{\xi^*} \theta) d^G Y_s + \frac{1}{2} \int_0^t \beta^* m \theta(d(\xi, Y_s), d(\xi, Y_s)).
\]

We claim that the $\beta_m(d(\xi, Y_t), d(\xi, Y_t))$ is null. In fact, take $0 \in T_g G$ and a left invariant vector field $Y$ on $G$. Here, $0$ is the vector associated to the constant
process $\xi$. Then
\begin{align*}
\beta_m(0, Y) &= \nabla^G_{m_*(0, Y)} m_*(0, Y) - m_* \nabla^{G \times G}(0, Y) \\
&= \nabla^G_{R_{h_*0 + L_{gs}(Y)}} (R_{h_*0 + L_{gs}(Y)}) - m_* \nabla^{G \times G}(0, Y) \\
&= \nabla^G_{L_{gs}Y} L_{gs}Y - L_{gs}(\nabla^G_Y Y) \\
&= L_{gs}(\nabla^G_Y Y) - L_{gs}(\nabla^G_Y Y) \\
&= 0,
\end{align*}
where in forth equality we use the fact that $\nabla^G$ is a left invariant connection. Thus we get
\[\int_0^t \theta d^G X_s = \int_0^t (L^*_\xi \theta) d^G Y_s.\]
As $Y_t$ is a solution of (5) we have
\[\int_0^t \theta d^G X_s = \int_0^t L^*_\xi(e) L^*_\xi(Y_s) \theta d^G M_s.\]
This gives
\[\int_0^t \theta d^G X_s = \int_0^t L^*_X(e) \theta d^G M_s.\]
Therefore we conclude that $X_t$ is a solution of (5).

The idea to show the existence and unicity of the solution of (5) is to construct a second order stochastic differential equations from this and use the results of the existence and unicity given by [7].

**Proposition 3.2** Let $M$ be a semimartingale in $g$ and $X_0$ a $F_0$-measurable random variable in $G$. There exist a predictable stopping time $\zeta$ and a $G$-valued semimartingale $X$ in $G$ on the interval $[0, \zeta]$, with initial condition $X_0$, solution to (5) and exploding to times $\zeta$ on the event $\{\zeta < \infty\}$. Moreover, the following uniqueness and maximality properties holds: if $\zeta'$ is a predictable time and $X'$ a solution starting from $X_0$ defined on $[0, \zeta']$, then $\zeta' \leq \zeta$ and $X' = X$ on $[0, \zeta']$.

**Proof:** From the Itô transfer principle, see Theorem 12 in [8], the stochastic differential equation (5) is equivalent to the intrinsic second order differential equation $d^2 X = f(X, M) d^2 M$, where $f: \tau_{M_t(\omega)} g \rightarrow \tau_{X_t(\omega)} G$ is the unique semi-affine Schwartz morphism with $L_{X_t(\omega)}(e)$ as restriction to the first order. From Lemma 11 in [8] we see that $f$ is a family of Schwartz morphism which depend smoothly upon $(e, g)$, for all $g \in G$. From Theorem 6.41 in [7], with a $F_0$-measurable random variables $X_0$ in $G$ as an initial condition, there exists
a predictable stopping time $\zeta > 0$ and a $G$-valued semimartingale $X$ on the interval $[0, \zeta]$ which is the solution of $d^2X = f(X, M)d^2M$ and exploding at time $\zeta$ on the event $\{\zeta < \infty\}$. Moreover, the uniqueness and maximality properties holds in the following sense: if $\zeta'$ is a predictable time and $X'$ is a solution starting from $X_0$ defined on $[0, \zeta']$, then $\zeta' \leq \zeta$ and $X' = X$ on $[0, \zeta']$. Again, Itô transfer principle assure the unique and maximal solution of (5) with initial condition $X_0$ and predictable stopping time $\zeta$. 

This Proposition deals with hypothesis that $\nabla^G$ and $\nabla^g$ are any connections on $G$ and $g$, respectively. However, if we wish study the martingales in $G$, we need some restriction in these connections. This restriction is that connections are completes as one see in the next proposition.

**Proposition 3.3** If $X_t$ is the solution of the stochastic differential equation (5), then this life time is infinity, since $\nabla^G$ and $\nabla^g$ are completes.

**Proof:** It is a direct consequence of Theorem 14 in [8] and the fact that every geodesic in $G$ and $g$ with respect to $\nabla^G$ and $\nabla^g$, respectively, are extended to all time. 

Proposition 3.1 says that we can consider the solution of stochastic differential equation (5) with initial value $X_0 = e$ rather than any random variable on $G$. In other side, Proposition 3.3 shows that solution of (5) is in interval $[0, \infty[$. From these facts we give the following definition.

**Definition 3.1** Suppose that $\nabla^G$ is a complete, left invariant connection on $G$ and $\nabla^g$ is a complete connection on $g$. We will denote by $e^{Gg}(M)$ the solution of (5) with initial condition $X_0 = e$ and we call it Itô stochastic exponential with respect to $\nabla^G$ and $\nabla^g$.

In the follow, for simplicity, we will call $e^{Gg}(M)$ by Itô exponential.

**Remark 3.2** It is well known that the left-invariant connections on $G$ are in one-one correspondence with bilinear forms on $g$, see Proposition 1, chapter 3 in [11]. However the stochastic differential equation (5) do not preserve this fact, that is, it is not necessary that $\nabla^G$ and $\nabla^g$ have some association.

As an immediate consequence from Proposition 3.2 we have a characterization of the $\nabla^G$-martingales in the Lie group $G$.

**Corollary 3.3** The Itô exponential $e^{Gg}(M)$ is a $\nabla^G$-martingale in $G$ if and only if $M$ is a $\nabla^g$-martingale in $g$.

The Itô exponential yields a semimartingale in $G$ from a semimartingale in $g$. M. Hakim-Dowek and D. Lépingle [10] define, in Stratonovich sense, an
inverse of the stochastic exponential, which they called the stochastic logarithm. We wish to get an analogous in the Itô sense. For this we consider a left-invariant connection $\nabla^G$ on $G$ and a connection $\nabla^\mathfrak{g}$ on $\mathfrak{g}$. Our idea is to create the process inverse of the Itô exponential as solution of the following stochastic differential equation

$$d^\mathfrak{g}M_t = L_{(X_t)^{-1}}(X_t)d^G X_t. \tag{6}$$

The solution to this stochastic differential equation means that for every 1-form $\psi$ on $\mathfrak{g}$ the real semimartingales $\int \psi d^\mathfrak{g}M$ and $\int L_{X_t}^{-1}(X_t)\psi d^G X$ are equal.

An important invariance property to the solution of stochastic differential equation above is obtained if we ask the left invariance property to $\nabla^G$.

**Proposition 3.4** Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Assume that $\nabla^G$ is a left-invariant connection on $G$ and $\nabla^\mathfrak{g}$ is a connection on $\mathfrak{g}$. Suppose that $M_t$ is a solution of (6) with respect to a semimartingale $Y_t$. If $\xi$ is a random variable in $G$, then $M_t$ is a solution of (6) with respect to $X_t = \xi Y_t$.

**Proof:** Let $\psi$ be a 1-form on $\mathfrak{g}$. By definition, the Itô stochastic differential equation (6) means that

$$\int_0^t \psi d^\mathfrak{g}M_s = \int_0^t L_{Y_s^{-1}}\psi d^G Y_s.$$ 

Since $Y_t = L_{\xi^{-1}}X_t$, it follows that

$$\int_0^t \psi d^\mathfrak{g}M_s = \int_0^t L_{(\xi^{-1}X_s)^{-1}}\psi d^G L_{\xi^{-1}}X_s.$$ 

As in the proof of Proposition 3.1 we have

$$\int_0^t \psi d^\mathfrak{g}M_s = \int_0^t L_{\xi^{-1}}^* L_{(\xi^{-1}X_s)^{-1}}\psi d^G X_s.$$ 

We thus get

$$\int_0^t \psi d^\nabla^\mathfrak{g}M_s = \int_0^t L_{X_s^{-1}}^* \psi d^\nabla^G X_s.$$ 

By definition, $M_t$ is also a solution of (6) with respect to $X_t$.

The solution and uniqueness to (6) are assured to follow.

**Proposition 3.5** Let $X$ be a semimartingale in $G$ and $M_0$ a $\mathcal{F}_0$-measurable random variable with value in $\mathfrak{g}$. There exists a predictable stopping time $\eta$ and a $\mathfrak{g}$-valued semimartingale $M$ in $\mathfrak{g}$ on the interval $[0, \eta]$, with initial condition $M_0$, solution to (6) and exploding to times $\eta$ on the event $\{\eta < \infty\}$. Moreover, the following uniqueness and maximality properties hold: if $\eta'$ is a predictable time and $M'$ a solution starting from $M_0$ defined on $[0, \eta']$, then $\eta' \leq \eta$ and $M' = M$ on $[0, \eta']$. 


Proof: The proof is analogous to one in Proposition 3.2. □

Proposition 3.6 The solution of (6) has a life time infinity, since $\nabla^G$ and $\nabla^g$ are completes.

Proof: The same argument that is used to prove Proposition 3.3. □

Propositions 3.4 - 3.6 yield the well definition of the Itô logarithm.

Definition 3.4 Suppose that $\nabla^G$ is a complete, left invariant connection and $\nabla^g$ is a complete connection. The solution to (6), with initial condition $M_0 = 0$, is called Itô stochastic logarithm with respect to $\nabla^G$ and $\nabla^g$ and it is denoted by $L^{G\theta}(X)$.

For simplicity, we call $L^{G\theta}(X)$ by Itô logarithm. Now, one may see, as in the case of the Itô exponential, the clear relation between $\nabla^g$-martingales and $\nabla^G$-martingales.

Corollary 3.5 A semimartingale $X_t$ in $G$ is a $\nabla^G$-martingale if and only if $L^{G\theta}(X)$ is a $\nabla^g$-martingale in $g$.

Our main intention to work with the Itô Logarithm is that it is the inverse of the Itô exponential.

Theorem 3.6 Let $\nabla^G$ be a complete, left invariant connection on $G$ and $\nabla^g$ a complete connection on $g$. Let $X_t, M_t$ be semimartingales in $G$ and $g$, respectively. Then

$$L^{G\theta}(e^{G\theta}(M_t)) = M_t$$

and

$$e^{G\theta}(L^{G\theta}(X_t)) = X_t.$$ 

Proof: Let $\psi$ be a 1-form on $g$ and $M_t$ a semimartingale in $g$. The Itô exponential is the semimartingale $e^{G\theta}(M_t)$ in $G$. By definition, the Itô logarithm apply to $e^{G\theta}(M_t)$ means that

$$\int_0^t \psi d^g L^{G\theta}(e^{G\theta}(M_s)) = \int_0^t L_{e^{G\theta}(M_s)}^{-1} \psi d^G e^{G\theta}(M_s).$$

Since $e^{G\theta}(M_t)$ is solution of (5), it follows that

$$\int_0^t \psi d^g L^{G\theta}(e^{G\theta}(M_s)) = \int_0^t L_{e^{G\theta}(M_s)} L_{e^{G\theta}(M_s)}^{-1} \psi d^g M_s = \int_0^t \psi d^g M_s.$$ 

As $\psi$ is an arbitrary 1-form on $g$ we have $L^{G\theta}(e^{G\theta}(M_t)) = M_t$. Similarly, for a semimartingale $X_t$ in $G$, we get $e^{G\theta}(L^{G\theta}(X_t)) = X_t$. □

To end this section, we characterize the martingales in Lie groups.
Theorem 3.7 Let $\nabla^G$ be a complete, left invariant connection on $G$ and $\nabla^g$ a complete connection on $g$. Every $\nabla^G$-martingale in $G$ is write as $e^{G\alpha}(M_t)$ for a $\nabla^g$-martingale $M_t$ in $g$.

Proof: Let $X_t$ be a $\nabla^G$-martingale in $G$. From Corollary 3.5 we see that $L^{G\alpha}(X_t)$ is a $\nabla^g$-martingale in $g$. Taking $M_t = L^{G\alpha}(X_t)$ we conclude that $X_t = e^{G\alpha}(M_t)$, which follows from Theorem 3.6.

4 Campbell-Hausdorff formulas

In this section, we give a stochastic Campbell-Hausdorff formula. For this end, we ask for a condition over the left invariant connection $\nabla^G$. More exactly, to the connection function $\alpha : g \times g \rightarrow g$ associated to $\nabla^G$ we suppose that $\alpha(A, A) = 0$ for all $A \in g$. It is known that $\nabla^G$ is complete. Before we show the stochastic Campbell-Hausdorff formulas we need to introduce the null quadratic variation property.

Definition 4.1 Let $N$ be a smooth manifold and $X_t, Y_t$ be two semimartingales in $N$. We say that $X_t$ and $Y_t$ have the null quadratic variation property if for any local coordinates system $(x^1, \ldots, x^n)$ on $N$ we have $[X^i_t, Y^j_t] = 0$, for $i, j = 1, \ldots, n$, where $X^i_t = x^i \circ X_t$ and $Y^j_t = y^j \circ Y_t$.

Example 4.2 Any two independents semimartingales in a smooth manifold $N$ have the null quadratic variation property.

The null quadratic variation property is good because it is held by the Itô logarithm.

Proposition 4.1 Let $G$ be a Lie group with a complete, left invariant connection $\nabla^G$ and $g$ its Lie algebra endowed with a complete connection $\nabla^g$. Given two semimartingales $X_t, Y_t$ in $G$, then $X_t, Y_t$ have the null quadratic variation property if and only if $L^{G\alpha}(X_t)$ and $L^{G\alpha}(Y_t)$ have the null quadratic variation property.

Proof: Suppose that $X_t, Y_t$ are two semimartingales in $G$ such that $X_t, Y_t$ have the null quadratic variation property. Thus, for any local coordinates system $(U, x^1, \ldots, x^n)$ on $G$ we see that $[X^i_t, Y^j_t] = 0$, where $X^i_t = x^i \circ X_t$ and $Y^j_t = x^j \circ Y_t$, $i, j = 1, \ldots, n$. It is sufficient to prove that $[L^{G\alpha}(X_t)^\alpha, L^{G\alpha}(Y_t)^\beta] = 0$ for a global coordinate system $(y^1, \ldots, y^n)$ on $g$. By Proposition 7.8 in [7],

$$[L^{G\alpha}(X_t)^\alpha, L^{G\alpha}(Y_t)^\beta] = \left[ \int_0^t dy^\alpha dL^{G\alpha}X_s, \int_0^t dy^\beta dL^{G\alpha}Y_s \right].$$
From definition of the Itô logarithm we see that

\[ [L^G_{\mathfrak{g}}(X_t)^\alpha, L^G_{\mathfrak{g}}(Y_t)^\beta] = \left[ \int_0^t dy^\alpha L^{-1}_{X_s}(X_s) d^G X_s, \int_0^t dy^\beta L^{-1}_{Y_s}(Y_s) d^G Y_s \right]. \]

Applying the definition of the integral of Itô (1) yields

\[ [L^G_{\mathfrak{g}}(X_t)^\alpha, L^G_{\mathfrak{g}}(Y_t)^\beta] = \left[ \sum_{l=1}^n \int_0^t (dy^\alpha L^{-1}_{X_s}(X_s))^l dX_s^l, \sum_{k=1}^n \int_0^t (dy^\beta L^{-1}_{Y_s}(Y_s))^k dX_s^k \right]. \]

Interchanging the integral of Itô with quadratic variation we obtain

\[ [L^G_{\mathfrak{g}}(X_t)^\alpha, L^G_{\mathfrak{g}}(Y_t)^\beta] = \sum_{l,k=1}^n \int_0^t (dy^\alpha L^{-1}_{X_s}(X_s))^l(dy^\beta L^{-1}_{Y_s}(Y_s))^k d[X_s^l, Y_s^k]. \]

Since \( X, Y \) are null quadratic variation, \( [L^G_{\mathfrak{g}}(X_t)^\alpha, L^G_{\mathfrak{g}}(Y_t)^\beta] = 0 \). It gives the null quadratic variation property for \( L^G_{\mathfrak{g}}(X) \) and \( L^G_{\mathfrak{g}}(Y) \).

Similarly, one can show that if \( L^G_{\mathfrak{g}}(X) \) and \( L^G_{\mathfrak{g}}(Y) \) have the null quadratic variation property, then \( X, Y \) also have the one property.

The Campbell-Hausdorff formula is then given by

**Theorem 4.3** Let \( G \) be a Lie group with a complete, left invariant connection \( \nabla^G \) such that its connection function \( \alpha \) satisfies \( \alpha(A, A) = 0 \) for all \( A \in \mathfrak{g} \) and \( \mathfrak{g} \) its Lie algebra endowed with a complete connection \( \nabla^\mathfrak{g} \). Given two semimartingales \( M, N \) in \( \mathfrak{g} \) which satisfy the null quadratic variation property, then

\[ e^{G_{\mathfrak{g}}}(M_t + N_t) = e^{G_{\mathfrak{g}}} \left( \int_0^t \text{Ad}(e^{G_{\mathfrak{g}}}(N_s)) dM_s \right) e^{G_{\mathfrak{g}}}(N_t). \] (7)

For two semimartingales \( X, Y \) in \( G \), which have the null quadratic variation property, we have

\[ L^G_{\mathfrak{g}}(X_t Y_t) = \int_0^t \text{Ad}(Y_s^{-1}) dL^G_{\mathfrak{g}}(X_s) + L^G_{\mathfrak{g}}(Y_t). \] (8)

**Proof:** We begin introducing the following notation

\[ X_t = e^{G_{\mathfrak{g}}} \left( \int_0^t \text{Ad}(e^{G_{\mathfrak{g}}}(N_s)) dM_s \right) \quad \text{and} \quad Y_t = e^{G_{\mathfrak{g}}}(N_t). \] (9)

The proof of (7) is complete if for each left invariant 1-form \( \theta \) on \( G \)

\[ \int_0^t \theta d^G(X_s Y_s) = \int_0^t \theta L_{(X_s Y_s)^*}(e) d(M_s + N_s). \]
Consider the product on Lie group as the application \( m : G \times G \rightarrow G \). Using the geometric Itô formula (4) we get

\[
\int_0^t \theta d^G(X_sY_s) = \int_0^t \theta d^G m(X_s, Y_s) = \int_0^t m^* \theta d^{G \times G}(X_s, Y_s) + \frac{1}{2} \int_0^t \beta_m^*(d(X_s, Y_s), d(X_s, Y_s)).
\]

We have \( \frac{1}{2} \int_0^t \beta_m^*(d(X_s, Y_s), d(X_s, Y_s)) = 0 \), because \( \alpha(A,A) = 0 \) for all \( A \in g \) and \( X,Y \) have the null quadratic variation property. Hence

\[
\int_0^t \theta d^G X_t Y_t = \int_0^t m^* \theta d^{G \times G}(X_s, Y_s).
\]

From Proposition 3.7 in [9] it may be conclude that

\[
\int_0^t \theta d^G(X_sY_s) = \int_0^t R^*_Y \theta d^GX_s + \int_0^t L^*_X \theta d^GY_s.
\]

Replacing (9) in this equality yields

\[
\int_0^t \theta d^G(X_sY_s) = \int_0^t R^*_Y \theta L^*_X(e) Ad(Y_s)dM_s + \int_0^t L^*_X \theta L^*_Y(e)dN_s.
\]

Here, an easy computation shows that

\[
\int_0^t \theta d^G(X_sY_s) = \int_0^t \theta L^*_X Y_s(e) dM_s + \int_0^t \theta L^*_Y X_s(e) dN_s
\]

and the proof is complete.

The equality (8) is a direct consequence of (7).

**Theorem 4.4** Under hypothesis of Theorem 4.3, if \( X_t, Y_t \) are \( \nabla^G \)-martingales in \( G \) with the null quadratic variation property, then \( X_t \cdot Y_t \) is a \( \nabla^G \)-martingale in \( G \).

**Proof:** Let \( X_t, Y_t \) be \( \nabla^G \)-martingales in \( G \). By Corollary 3.5, it is sufficient to show that \( \mathcal{L}^{G\theta}(X_t \cdot Y_t) \) is a \( \nabla^\theta \)-martingale.

From Theorem 4.3 we see that

\[
\mathcal{L}^{G\theta}(X_t \cdot Y_t) = \int_0^t Ad(Y_s^{-1})d\mathcal{L}^{G\theta}(X_s) + \mathcal{L}^{G\theta}(Y_t).
\]
Since $X_t, Y_t$ are $\nabla^g$-martingales, Corollary 3.5 assures that $\mathcal{L}^g(X_t)$ and $\mathcal{L}^g(Y_t)$ are $\nabla^g$-martingales. Also, $\int_0^t \text{Ad}(Y_s^{-1})d\mathcal{L}^g(X_s)$ is a $\nabla^g$-martingale because for any $1$-form $\theta$ on $g$ we have

$$\int_0^t \theta d^g\text{Ad}(Y_r^{-1})d^g\mathcal{L}^g(X_r) = \int_0^t \theta \text{Ad}(Y_s^{-1})d^g\mathcal{L}^g(X_s) = \int_0^t (\text{Ad}(Y_s^{-1})^*\theta)d^g\mathcal{L}^g(X_s),$$

Thus the sum of right side of (10) yields a $\nabla^g$-martingale and, consequently, the proof is complete. \hfill $\square$ In follows, we generalize the result due to P. Catuogno and P. Ruffino [4] for product of harmonic maps.

Before, we are going to introduce a stochastic characterization for harmonic maps. Let $(M, g)$ be a Riemannian manifold and $N$ a smooth manifold with a connection $\nabla^N$. A smooth map $F : M \to N$ is a harmonic map if and only if it sends $g$-Brownian motions to $\nabla^N$-martingales.

**Proposition 4.2** Let $(M_j, g_j)$, $j = 1, \ldots, n$, be Riemannian manifolds, $G$ a Lie group with a complete, left-invariant connection $\nabla^G$ such that $\alpha(A, A) = 0$ for all $A \in g$ and $g$ its Lie algebra endowed with a complete connection $\nabla^g$. If $\phi_j : (M_j, g_j) \to (G, \nabla^G)$ are harmonic maps, then the product map $\phi_1 \cdot \phi_2 \cdots \phi_n$ from $M_1 \times M_2 \times \ldots \times M_n$ into $G$ is a harmonic map.

**Proof:** It is enough to take $n = 2$. Let $\phi_1 : (M_1, g_1) \to (G, \nabla^G)$ and $\phi_2 : (M_2, g_2) \to (G, \nabla^G)$ be harmonic maps. Let $B^1_t$ and $B^2_t$ two independent Brownian motions in $M_1$ and $M_2$, respectively. Thus $(B^1_t, B^2_t)$ is a Brownian motion in the Riemannian product manifold $M_1 \times M_2$. It is sufficient to show that $\phi_1(B^1_t) \cdot \phi_2(B^2_t)$ is a $\nabla^G$-martingale. Since $B^1_t$ and $B^2_t$ are independent, $\phi_1(B^1_t)$ and $\phi_2(B^2_t)$ are too. Consequently, they have the null quadratic variation. Being $\phi_1(B^1_t)$ and $\phi_2(B^2_t)$ $\nabla^G$-martingales, Theorem 4.4 now shows that $\phi_1(B^1_t) \cdot \phi_2(B^2_t)$ is a $\nabla^G$-martingale, and the proof is complete. \hfill $\square$

**Example 4.5** Let $G$ be a Lie group with a bi-invariant metric and $g$ its Lie algebra with the Levi-Civita connection $\nabla^G$. Seeing the product on $G$ as application $m : (G \times G, \nabla^G \times \nabla^G) \to (G, \nabla^G)$, Theorem 4.4 shows that the product $m$ is harmonic map. \hfill $\square$

**Example 4.6** Let $G$ be a Lie group equipped with a left-invariant connection $\nabla^G$ such that its connection function $\alpha$ satisfies $\alpha(A, A) = 0$ for all $A \in g$ and $g$ its Lie algebra endowed with a complete connection $\nabla^g$. Let $\gamma_i$ be $\nabla^G$-geodesics in $G$, $i = 1, \ldots, n$. A map $f : (\mathbb{R}^n, <,>) \to G$ defined by

$$f(t_1, t_2, \ldots, t_n) = \gamma_1(t_1) \cdot \gamma_1(t_2) \cdot \cdots \gamma_n(t_n).$$

is harmonic. It is a direct consequence of Theorem 4.2. Indeed, it is sufficient to see any geodesic $\gamma_i$, $i = 1, \ldots, n$, is a harmonic map.
In particular, assume that $G$ has a bi-invariant metric. Choose $n$ vectors $X_1, X_2, \ldots, X_n \in G$ such that $\exp(t_1 X_1), \exp(t_2 X_2), \ldots, \exp(t_n X_n)$ are geodesics (see for instant [6]). According to the facts above, $\exp(t_1 X_1) \cdot \exp(t_2 X_2) \cdot \ldots \cdot \exp(t_n X_n)$ is a harmonic map. This example is also founded in [6] and [4].

5 Itô logarithm and examples

We begin recalling the definition of the stochastic logarithm due to M. Hakim-Dowek and D. Lépine [10]. Let $X_t$ be a semimartingale in $G$, then the stochastic logarithm, denoted by $L(X_t)$, is the solution of the stochastic differential equation

$$\delta L(X_t) = L(X_t)^{-1}(X_t)\delta X_t$$

and $X_0 = e$.

Denoting by $\omega$ the Maurrer-Cartan form on $G$, it is simple to see that $L(X_t) = \int_0^t \omega \delta X_s$. Similarly, taking in account that the Itô logarithm is a solution to stochastic differential equation (6) we get $L^G(X_t) = \int_0^t \omega dL^G X_s$.

Following, we give a relation between the stochastic logarithm and Itô logarithm.

Proposition 5.1 Let $G$ be a Lie group with a complete, left invariant connection $\nabla^G$ with the associated connection function $\alpha$ and $\mathfrak{g}$ its Lie algebra endowed with a complete connection $\nabla^\mathfrak{g}$. A semimartingale $X_t$ in $G$ is a $\nabla^G$-martingale if and only if

$$L(X_t) + \frac{1}{2} \int_0^t \alpha(L(X_s), L(X_s)).$$

is a local martingale in $\mathfrak{g}$.

Proof: Let $X_t$ be a semimartingale in $G$. From the Itô-Stratonovich formula of conversion (2) we compute

$$\mathcal{L}^\mathfrak{g}(X_t) = \int_0^t \omega_G d\nabla^G X_s = \int_0^t \omega_G \delta X_s + \frac{1}{2} \int_0^t \nabla^G \omega(dX_s, dX_s)$$

$$= \int_0^t \omega_G \delta X_s + \frac{1}{2} \int_0^t \alpha(\omega dX_s, \omega dX_s)$$

$$= L(X_t) + \frac{1}{2} \int_0^t \alpha(L(X_s), L(X_s)).$$

Therefore the proof follows from Corollary 3.5.

The principal significance of this Proposition is that it allows us to see the geometry related whit martingales in Lie groups. Specifically, $\frac{1}{2} \int_0^t \alpha(L(X_s), L(X_s))$ is the term that differentiate the martingales in Lie Groups in accord to geometry given by the connection $\nabla^G$. We see this fact in the next two example.
Example 5.1 If $G$ has a bi-invariant metric, then the Levi-Civita connection on $G$ is given by $(\nabla^G_A B)(e) = \frac{1}{2}[A, B]$, where $A, B \in \mathfrak{g}$. So the connection function $\alpha$ associate to $\nabla^G$ is $\alpha(A, B) = \frac{1}{2}[A, B]$. Therefore, $\alpha(A, A) = 0$.

We thus conclude that $X_t$ is a $\nabla^G$-martingale if and only if $L(X_t)$ is a local martingale in $\mathfrak{g}$.

A direct application of this is any semisimple Lie group $G$ equipped with the metric given by the Killing form. \hfill $\square$

Example 5.2 Suppose that $G$ is equipped with a complete left-invariant metric. Then the Levi-Civita connection on $G$ is given by $(\nabla^G_A B)(e) = \frac{1}{2}[A, B] + U(A, B)$, where $A, B \in \mathfrak{g}$ and $U: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear mapping defined by

$$2 < U(A, B), C > = < A, [C, B] > + < [C, A], B >,$$

for all $A, B, C \in \mathfrak{g}$.

Here, $<,>$ is the scalar product on $\mathfrak{g}$ associated to the left metric on $G$. It follows that the connection function $\alpha$ associate to $\nabla^G$ is given by $\alpha(A, B) = \frac{1}{2}[A, B] + U(A, B)$. Consequently, $\alpha(A, A) = U(A, A)$. We thus conclude that $X_t$ is a $\nabla^G$-martingale if and only if $L(X_t) + \frac{1}{2} \int_0^t U(L(X_s), L(X_s))$ is a local martingale in $\mathfrak{g}$. \hfill $\square$

In the sequel, we study the martingales in some matrix Lie groups with a left invariant metric. The idea is based in the work [6], where it describes the bilinear mapping $U$ for some specific Lie groups. This fact and example above allows to characterize the martingales in these Lie groups. We begin with the Euclidian motion group $SE(3)$ and, in the sequel, with the three-dimensional non-compact Lie groups $SE(2), E(1, 1), N^3$ and $SL(2, \mathbb{R})$.

Example 5.3 (Euclidian motion group $SE(3)$) The Euclidian motion group $SE(3)$ is defined by

$$SE(3) = \left\{ (X, u) = \begin{pmatrix} X & u \\ 0 & 0 \end{pmatrix}; X \in SO(3), u \in \mathbb{R}^3 \right\},$$

where $0$ is the $1 \times 3$ matrix consisting of $0$ and $u$ is a $3$-column vector in $\mathbb{R}^3$. The Lie algebra $\mathfrak{se}(3)$ is given by

$$\mathfrak{se}(3) = \left\{ (X, u) = \begin{pmatrix} X & u \\ 0 & 0 \end{pmatrix}; X \in \mathfrak{so}(3), u \in \mathbb{R}^n \right\}.$$

For our study we consider the inner product $<,>_\lambda$, with $\lambda > 0$, defined by

$$< (A, x), (B, y) >_\lambda = -\frac{1}{4} \text{tr}(AB) + \lambda x^t y,$$

where $x^t$ is the transpose of $x$. Take the basis $\beta = \{E_1, E_2, E_3, e_1, e_2, e_3\}$ of $\mathfrak{se}(3)$, where
$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $X_t$ be a semimartingale in $SE(3)$ and $L(X_t)$ the stochastic logarithm in $\text{se}(3)$. According to basis $\beta$, we may write $L(X_t) = \sum_{i=1}^3 x_i(t)E_i + \sum_{i=1}^3 y_i(t)e_i$, where $x_i(t), y_i(t)$ are real semimartingales. Then from Lemma 5.1 in [6] we see that $U(L(X_t), L(X_t)) = x(t) \times y(t)$ (the vector product), where

$$x(t) = \sum_{i=1}^3 x_i(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \in \mathbb{R}^3 \text{ and } y(t) = \sum_{i=1}^3 y_i(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} \in \mathbb{R}^3.$$

From Example 5.2 we see that $X_t$ is a $\nabla^G$-martingale if and only if

$$L(X_t) + \frac{1}{2} \int_0^t x(s) \times y(s)$$

is a local martingale in $\text{se}(3).$ \hfill $\square$

**Example 5.4 (SE(2))** With a little modification of the example above, we have as a basis for the Lie algebra $\text{se}(2)$

$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The inner product adopt $<, >_\lambda$, with $\lambda > 0$, in $\text{se}(2)$ is defined by

$$< (aH + x_1 e_1 + x_2 e_2), (bH + y_1 e_1 + y_2 e_2) >= ab + \lambda^2 (x_1 y_1 + x_2 y_2).$$

Let $X_t$ be a semimartingale in $SE(2)$ and $L(X_t)$ the stochastic logarithm in $\text{se}(2)$. We may write $L(X_t) = a(t)H + a_1(t)e_1 + a_2(t)e_2$, where $a(t), a_1(t), a_2(t)$ are real semimartingales. Lemma 6.1 in [6] now assures that $U(L(X_t), L(X_t)) = a(t)H(a_1(t)e_1 + a_2(t)e_2)$. We conclude from Example 5.2 that $X_t$ is a $\nabla^G$-martingale if and only if

$$L(X_t) + \frac{1}{2} \int_0^t a(s)H(a_1(s)e_1 + a_2(s)e_2)$$

is a local martingale in $\text{se}(2).$ \hfill $\square$
Example 5.5 (E(1,1)) The three-dimensional Lie group $E(1, 1)$ is given by

$$E(1, 1) = \left\{ \begin{pmatrix} \exp(\xi) & 0 & x_1 \\ 0 & \exp(-\xi) & x_2 \\ 0 & 0 & 1 \end{pmatrix}; \xi, x_1, x_2 \in \mathbb{R} \right\},$$

with standard multiplication of matrix. Its Lie algebra $\mathfrak{e}(1, 1)$ is given by

$$\mathfrak{e}(1, 1) = \left\{ \begin{pmatrix} \xi & 0 & x_1 \\ 0 & -\xi & x_2 \\ 0 & 0 & 1 \end{pmatrix}; \xi, x_1, x_2 \in \mathbb{R} \right\}. $$

A basis for $\mathfrak{e}(1, 1)$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The inner product $<, >_{\lambda}(\lambda > 0)$ adopted on $\mathfrak{e}(1, 1)$ is

$$< (aH + x_1e_1 + x_2e_2), (bH + y_1e_1 + y_2e_2) >_{\lambda} = ab + \lambda^2(x_1y_1 + x_2y_2).$$

Take a semimartingale $X_t$ in $E(1, 1)$ and the stochastic logarithm $L(X_t)$ in $\mathfrak{e}(1, 1)$. So the stochastic logarithm may be written as $L(X_t) = a(t)H + a_1(t)e_1 + a_2(t)e_2$. Then from Lemma 6.4 in [6] we see that $U(L(X_t), L(X_t)) = \|a_1(t)e_1 + a_2(t)e_2\|^2 \lambda^2 H - a(t)H(a_1(t)e_1 + a_2(t)e_2)$, where $\|a_1(t)e_1 + a_2(t)e_2\|^2 = a_1^2(t) - a_2^2(t)$. Therefore, Example 5.2 assures that $X_t$ is a $\nabla^G$-martingale if and only if

$$L(X_t) + \frac{1}{2} \int_0^t \|a_1(s)e_1 + a_2(s)e_2\|^2 \lambda^2 H - a(s)H(a_1(s)e_1 + a_2(s)e_2)$$

is a local martingale in $\mathfrak{e}(1, 1)$.

Example 5.6 (N$^3$) The Heisenberg group is the three dimensional nilpotent group defined by

$$N^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\}.$$ 

The Lie algebra $\mathfrak{n}^3$ of the Heisenberg group is

$$\mathfrak{n}^3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}; x, y, z \in \mathbb{R} \right\}.$$
A basis for \( n^3 \) is

\[
X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

We adopt the inner product \(<, >_\lambda (\lambda > 0)\) on \( n^3 \) given by

\[
< (a_1 X + b_1 Y + c_1 Z), (a_2 X + b_2 Y + c_2 Z) > = a_1 a_2 + \lambda^2 (b_1 b_2 + c_1 c_3).
\]

Let \( X_t \) be a semimartingale in \( N^3 \) and \( L(X_t) \) the stochastic logarithm in \( n^3 \). Write \( L(X_t) = a(t)H + b(t)E_+ + c(t)E_- \) in terms of basis above. Using Lemma 6.9 in [6] we obtain \( U(L(X_t), L(X_t)) = \lambda^2 b(t)c(t)X + \lambda^2 a(t)c(t)Y \). So Example 5.2 shows that \( X_t \) is a \( \nabla^G \)-martingale if and only if

\[
L(X_t) + \frac{1}{2} \int_0^t -\lambda^2 b(s)c(s)X + \lambda^2 a(s)c(s)Y
\]

is a local martingale in \( n^3 \).

**Example 5.7 (SL(2, \mathbb{R}))** The Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \) of \( SL(2, \mathbb{R}) \) is \( \{H, E_+, E_-\} \), where

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Assume that \( \mathfrak{sl}(2, \mathbb{R}) \) is equipped with the inner product \(<, >_\lambda (\lambda > 0)\) given by

\[
< (a_1 H + b_1 E_+ + c_1 E_-), (a_2 H + b_2 E_+ + c_2 E_-) > = a_1 a_2 + \lambda^2 (b_1 b_2 + c_1 c_3).
\]

Taking a semimartingale \( X_t \) in \( SL(2, \mathbb{R}) \) and the stochastic logarithm \( L(X_t) \) in \( \mathfrak{sl}(2, \mathbb{R}) \) we have \( L(X_t) = a(t)X + b(t)Y + c(t)Z \), where \( a(t), b(t), c(t) \) are real semimartingales. Applying Lemma 6.4 in [6] we see that

\[
U(L(X_t), L(X_t)) = \frac{2}{\lambda^2} b(t)^2 - c(t)^2 \lambda E_+ + (-2a(t)b(t) + a(t)c(t)\lambda)E_+ + (-a(t)b(t)\lambda + 2a(t)c(t))E_-.
\]

Now, Example 5.2 assures that \( X_t \) is a \( \nabla^G \)-martingale if and only if

\[
L(X_t) + \frac{1}{2} \int_0^t \frac{2}{\lambda^2} (b(s)^2 - c(s)^2)H + (-2a(s)b(s) + a(s)c(s)\lambda)E_+ + (-a(s)b(s)\lambda + 2a(s)c(s))E_-
\]

is a local martingale in \( \mathfrak{sl}(2, \mathbb{R}) \).
References


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