Classical Matrices and Mean-Starshaped Sequences

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Dedicated to Dr. Fred Leeds on his retirement

Abstract

We give the necessary and sufficient conditions for a lower triangular matrix to preserve the mean-starshape of sequences. Further, we show that the Cesaro matrix, Euler matrix and a particular case of Nörlund matrix preserve the mean-starshape of sequences.

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1. Introduction. In [5], Toader introduced the notion of starshaped sequences and mean-starshaped sequences and showed that the sets of convex and mean-convex sequences are subsets of mean-starshaped sequences. Several authors [1, 2, 4, 5] have proved many results on convex sequences and mean-convex sequences. We hardly find any study on mean-starshaped sequences. In [6] and [7], Toader proved a few results on \( u \)-mean starshaped sequences and \( p \)-mean starshaped sequences. In this paper, we prove the results on the transformations of mean-starshaped sequences by a lower triangular matrix \( A \). We also prove several relationships between starshaped...
and mean-starshaped sequences. We begin with the following definitions given in [5].

**Definition 1.1.** A real sequence \( \{x_n\} \) is called starshaped, if

\[
S(x_n) = \frac{x_n - x_0}{n} - \frac{x_{n-1} - x_0}{n-1} \geq 0 \quad \text{for} \quad n \geq 2.
\] (1.1)

**Definition 1.2.** A real sequence \( \{x_n\} \) is called mean-starshaped, if its mean-sequence denoted by \( \mathcal{M}(x_n) \) where

\[
\mathcal{M}(x_n) = \frac{x_0 + \cdots + x_n}{n+1}
\]

is starshaped. In other words, \( \{x_n\} \) is mean-starshaped if

\[
S(\mathcal{M}(x_n)) = \frac{\mathcal{M}(x_n) - \mathcal{M}(x_0)}{n} - \frac{\mathcal{M}(x_{n-1}) - \mathcal{M}(x_0)}{n-1} \geq 0, \quad \text{for} \quad n \geq 2.
\] (1.2)

It is easy to see that the operators \( \mathcal{M} \) and \( S \) are linear. So, we state below the lemma without proof.

**Lemma 1.3.** If \( a \) and \( b \) are arbitrary real numbers and if \( \{x_n\} \) and \( \{y_n\} \) are real sequences, then

(i) \( \mathcal{M}(ax_n + by_n) = a\mathcal{M}(x_n) + b\mathcal{M}(y_n) \).

(ii) \( S(ax_n + by_n) = aS(x_n) + bS(y_n) \).

(iii) If \( a \) and \( b \) are non-negative real numbers and \( S(x_n) \geq 0 \) and \( S(y_n) \geq 0 \), then \( S(ax_n + by_n) \geq 0 \).

Also, we give below a relationship between \( S(x_n) \) and \( S(\mathcal{M}(x_n)) \), which we will use later in the paper.

For any real sequence \( \{x_n\} \), we have from the definitions of \( S \) and \( \mathcal{M} \),

\[
S(\mathcal{M}(x_n)) = \frac{\mathcal{M}(x_n) - \mathcal{M}(x_0)}{n} - \frac{\mathcal{M}(x_{n-1}) - \mathcal{M}(x_0)}{n-1} = \frac{1}{n} \left[ \frac{x_0 + x_1 + \cdots + x_n}{n + 1} - x_0 \right] - \frac{1}{n-1} \left[ \frac{x_0 + x_1 + \cdots + x_{n-1}}{n} - x_0 \right]
\]

\[
= \frac{-2}{n(n^2 - 1)} \sum_{k=0}^{n-1} x_k + \frac{x_n}{n(n+1)} + \frac{x_0}{n(n-1)}
\]

\[
= \frac{1}{n(n^2 - 1)} \left[ (n-1)x_n - 2(x_{n-1} + \cdots + x_1) + (n-1)x_0 \right]
\]

\[
= \frac{1}{n(n^2 - 1)} \sum_{k=2}^{n} [ (k-1)x_k - kx_{k-1} + x_0 ]
\] (1.3)
The result given below involving the inequalities of $S(x_n)$ and $S(M(x_n))$ follows easily from the above relationship between $S(x_n)$ and $S(M(x_n))$. Hence, we omit the proof.

**Lemma 1.4.** For any real sequence $\{x_n\}$, if $S(x_n) \geq 0$ for each $n \geq 2$, then $S(M(x_n)) \geq 0$.

**Lemma 1.5.** For any real sequence $\{x_n\}$, $S(x_n) = 0$ if and only if $S(M(x_n)) = 0$ for each $n \geq 2$.

**Proof.** From (1.4), it is obvious that $S(x_n) = 0$ implies $S(M(x_n)) = 0$. We prove the converse by induction. Since $S(M(x_2)) = 0$, we have from (1.4) that $S(M(x_2)) = \frac{1}{3}S(x_2) = 0$. Now, assuming that $S(x_k) \geq 0$ for $k = 2, 3, \ldots, n$, we will show that $S(x_{n+1}) \geq 0$. Since $S(M(x_{n+1})) = 0$, using (1.4), we can write

$$
\frac{1}{(n+1)(n^2+2n)} \sum_{k=2}^{n+1} k(k-1)S(x_k) = 0.
$$

By our assumption, the above equation simplifies to

$$
\frac{1}{(n+2)}S(x_{n+1}) = 0.
$$

Hence, the lemma.

In the next section, we give a few more definitions and notations. In Section 3, we state and prove the necessary and sufficient conditions for a lower triangular matrix to map a mean-starshaped sequence into a mean-starshaped sequence. Finally, in Section 4, we discuss the classical matrices.

**2. Preliminary results.** In [5], Toader states the following result.

**Lemma 2.1.** If the sequence $\{x_n\}$ is represented by

$$
x_0 = c_0; \quad x_1 = -c_0 + 2c_1; \quad x_n = (1-2n)c_0 + 2n \sum_{j=1}^{n-1} \frac{c_j}{j} + (n+1)c_n \text{ for } n \geq 2, \quad (2.1)
$$

then $\{x_n\}$ is mean-starshaped if and only if $c_k \geq 0$ for $k \geq 2$. 

Proof. It suffices to show that $S(M(x_n)) \geq 0$ if and only if $c_k \geq 0$ for $k \geq 2$.

From (1.3), we have for $n \geq 2$

$$S(M(x_n)) = \frac{1}{n(n^2 - 1)} \left[(n - 1)x_n - 2(x_{n-1} + \cdots + x_1) + (n - 1)x_0\right].$$

Expressing $x_i$'s in terms of $c_i$'s as in (2.1), we have

$$S(M(x_n)) = \frac{1}{n(n^2 - 1)} \left[(n - 1) \left\{ (1 - 2n)c_0 + 2n \left( c_1 + \frac{c_2}{2} + \cdots + \frac{c_{n-2}}{n-2} + \frac{c_{n-1}}{n-1} \right) + (n + 1)c_n \right\}
- 2 \left\{ (3 - 2n)c_0 + 2(n - 1) \left( c_1 + \frac{c_2}{2} + \cdots + \frac{c_{n-2}}{n-2} \right) + nc_{n-1} \right\}
\vdots
- 2 \left\{ (-3)c_0 + 2 \cdot 2c_1 + 3c_2 \right\}
- 2 \left\{ (-1)c_0 + 2c_1 \right\}
+ (n - 1)c_0 \right].$$

Now, combining the coefficients of $c_k$'s we obtain

$$S(M(x_n)) = \frac{1}{n(n^2 - 1)} \left[ c_0 \left\{ (n - 1)(1 - 2n) + 2 \left( 1 + 3 + 5 + \cdots + (2n - 3) \right) + (n - 1) \right\}
+ 2 \sum_{k=1}^{n-2} \frac{c_k}{k} \left\{ n(n - 1) - 2 \left( (n - 1) + (n - 2) + \cdots + (k + 1) \right) - k(k + 1) \right\}
+ 2 \frac{c_{n-1}}{n-1} \left\{ n(n - 1) - (n - 1)n \right\}
+ c_n(n^2 - 1) \right].$$

A simple calculation shows that the coefficients of $c_0, c_1, \ldots, c_{n-1}$ are zero. Thus,

$$S(M(x_n)) = \frac{1}{n(n^2 - 1)} c_n(n^2 - 1) = \frac{1}{n} c_n, \text{ for } n \geq 2.$$ 

Hence, the result.

Also, it is easy to see that in the representation (2.1), the corresponding sequence \{c_k\} satisfies

$$c_0 = x_0, \quad c_1 = \frac{1}{2}(x_0 + x_1),$$

and for $k \geq 2$,

$$c_k = \frac{1}{k + 1} x_0 - \frac{2}{k^2 - 1} \sum_{j=1}^{k-1} x_j + \frac{1}{k + 1} x_k. \quad (2.2)$$
Definition 2.2. Let $A$ be a lower-triangular matrix defining a sequence-to-sequence transformation by

$$(Ax)_n = \sum_{k=0}^{n} a_{n,k} x_k.$$ 

We define the corresponding lower triangular matrices $A^{(1)} = [a_{n,k}^{(1)}]$, $A^{(2)} = [a_{n,k}^{(2)}]$, and $A^{(*)} = [a_{n,k}^{(*)}]$ as follows.

$$a_{n,i}^{(1)} = \sum_{k=i}^{n} a_{n,k}, \quad a_{n,i}^{(2)} = \sum_{k=i}^{n} a_{n,k}, \quad \text{and} \quad (2.3)$$

$$a_{n,i}^{(*)} = (i + 1)a_{n,i} + 2a_{n,i+1} + \left(\frac{2}{i}\right)a_{n,i+1},$$

for $i = 0, 1, \cdots, n$. Thus,

$$a_{n,i}^{(2)} = \sum_{k=i}^{n} \left( \sum_{j=k}^{n} a_{n,j} \right) = \sum_{j=i}^{n} (j - i + 1)a_{n,j}$$

and

$$a_{n,i}^{(*)} = (i + 1)a_{n,i} + \left(\frac{2}{i}\right) \sum_{k=i+1}^{n} k a_{n,k}. \quad (2.4)$$

Also, it is easy to see that

$$a_{n,n}^{(2)} = a_{n,n}^{(1)} = a_{n,n} \quad \text{and} \quad a_{n,n}^{(*)} = (n + 1)a_{n,n}. \quad (2.5)$$

Throughout this paper, the operators $\mathcal{M}$ and $S$ are applied to the column sequences of the above mentioned matrices.

For any fixed $i$, let

$$\mathcal{M}(a_{n,i}) = \frac{a_{0,i} + a_{1,i} + \cdots + a_{n,i}}{n+1},$$

$$\mathcal{M}(a_{n,i}^{(j)}) = \frac{a_{0,i}^{(j)} + a_{1,i}^{(j)} + \cdots + a_{n,i}^{(j)}}{n+1} \quad \text{for } j = 1, 2,$$

$$\mathcal{M}(a_{n,i}^{(*)}) = \frac{a_{0,i}^{(*)} + a_{1,i}^{(*)} + \cdots + a_{n,i}^{(*)}}{n+1}.$$

Since $A^{(*)} = [a_{n,k}^{(*)}]$ is lower triangular, $a_{n,i}^{(*)} = 0$ if $n < i$. Therefore, for $i \geq 1$

$$S(\mathcal{M}(a_{n,i})_n) = \frac{\mathcal{M}(a_{n,i}) - \mathcal{M}(a_{0,i})}{n} - \frac{\mathcal{M}(a_{n-1,i}) - \mathcal{M}(a_{0,i})}{n-1}.$$
Proof. Lemma 2.3. If

\[ \mathcal{M}(a_{n,i}^{(*)}) = \frac{a_{n,i}^{(*)}}{n} \]

Then

\[ S(\mathcal{M}(a_{n,i}^{(*)})) = \frac{\mathcal{M}(a_{n,i}^{(*)})}{n} - \frac{\mathcal{M}(a_{n-1,i}^{(*)})}{n-1} \]

where

\[ \mathcal{M}(a_{n,i}^{(*)}) = \frac{a_{i,i}^{(*)} + \cdots + a_{n,i}^{(*)}}{n+1}. \]

Also, from (2.5)

\[ \mathcal{M}(a_{n,n}^{(*)}) = \frac{a_{n,n}^{(*)}}{n+1} = a_{n,n}. \] (2.6)

Therefore, for \( n \geq 2 \),

\[
S(\mathcal{M}(a_{n,i}^{(*)})) = \frac{1}{n} \left( \frac{a_{i,i}^{(*)} + \cdots + a_{n,i}^{(*)}}{n+1} \right) - \frac{1}{n-1} \left( \frac{a_{i,i}^{(*)} + \cdots + a_{n-1,i}^{(*)}}{n} \right)
\]

\[
= \frac{1}{n(n^2-1)} \left[ -2 \left( a_{i,i}^{(*)} + \cdots + a_{n-1,i}^{(*)} \right) \right] + \frac{1}{n(n+1)} a_{n,i}^{(*)}
\]

\[
= \frac{1}{n(n^2-1)} \sum_{m=i}^{n} \left( (m-1)a_{m,i}^{(*)} - ma_{m-1,i}^{(*)} \right) \text{ where } a_{i-1,i} = 0 \text{ (2.7)}
\]

By the linearity of the operators \( S \) and \( \mathcal{M} \), the following result is obvious.

**Lemma 2.3.** If \( a_{n,i}^{(*)} \) is represented by (2.3), then for each fixed \( i \),

(i) \( \mathcal{M}(a_{n,i}^{(*)}) = (i+1)\mathcal{M}(a_{n,i}) + 2\mathcal{M}(a_{n,i+1}) + \left( \frac{2}{n} \right) \mathcal{M}(a_{n,i+1}) \)

(ii) \( S(\mathcal{M}(a_{n,i}^{(*)})) = (i+1)S(\mathcal{M}(a_{n,i})) + 2S(\mathcal{M}(a_{n,i+1})) + \left( \frac{2}{n} \right) S(\mathcal{M}(a_{n,i+1})) \).

Also, we need the following two lemmas to prove the main result.

**Lemma 2.4.** For each of the \( i \)-th column of the matrix \( A \), if \( S(a_{n,i}) \geq 0 \) for \( n \geq 2 \), then

(i) \( S(a_{n,i}^{(1)}) \geq 0; \ S(a_{n,i}^{(2)}) \geq 0; \ S(a_{n,i}^{(*)}) \geq 0; \)

(ii) \( S(\mathcal{M}(a_{n,i}^{(1)})) \geq 0; \ S(\mathcal{M}(a_{n,i}^{(2)})) \geq 0; \ S(\mathcal{M}(a_{n,i}^{(*)})) \geq 0. \)

**Proof.** By the linearity of \( S \), we can write

\[
S(a_{n,i}^{(1)}) = S \left( \sum_{k=i}^{n} a_{n,k} \right) = \sum_{k=i}^{n} S(a_{n,k}) \geq 0 \quad \text{and}
\]

\[
S(a_{n,i}^{(2)}) = S \left( \sum_{j=i}^{n} (j-i+1)a_{n,i} \right) = \sum_{j=i}^{n} (j-i+1)S(a_{n,j}) \geq 0.
\]

Similarly, \( S(a_{n,i}^{(*)}) \geq 0. \) This proves part (i) and part (ii) follows from Lemma 1.4.
Lemma 2.5. For each of the \( i \)-th column of the matrix \( A \), if \( S(a_{n,i}) = 0 \) for \( n \geq 2 \), then by the linearity of \( S \) and \( M \)

(i) \( S(a^{(1)}_{n,i}) = 0; \ S(a^{(2)}_{n,i}) = 0; \ S(a^{(e)}_{n,i}) = 0; \)

(ii) \( S(M(a^{(1)}_{n,i})) = 0; \ S(M(a^{(2)}_{n,i})) = 0; \ S(M(a^{(e)}_{n,i})) = 0. \)

It is not hard to see that the conditions (i) and (ii) do not imply that \( S(a_{n,i}) = 0 \) for each \( i \).

3. Main results. We give below the necessary and sufficient conditions for a lower triangular matrix to preserve mean-starshape of the sequences.

Theorem 3.1. A lower triangular matrix \( A \) preserves mean-starshape of sequences if and only if for each \( n = 2, 3, \ldots \)

(i) \( S(M(a^{(1)}_{n,0})) = 0, \)

(ii) \( S(M(a^{(e)}_{n,1})) = 0, \)

(iii) \( S(M(a^{(e)}_{n,k})) \geq 0, \) for \( k \geq 2. \)

Proof. Let \( \{x_n\} \) be a mean-starshaped sequence. Then by the representation (2.1)

\[
x_0 = c_0; \quad x_1 = -c_0 + 2c_1; \quad x_n = (1 - 2n)c_0 + 2n \sum_{j=1}^{n-1} j + (n + 1)c_n, \quad \text{for} \ n \geq 2,
\]

we have \( c_k \geq 0 \) for \( k \geq 2. \) Then the \( n \)-th term of the transformed sequence is

\[
(Ax)_n = \sum_{k=0}^{n} a_{n,k} x_k
\]

\[
= a_{n,0}c_0 + a_{n,1}(2c_1 - c_0) + \sum_{k=2}^{n} a_{n,k} \left( (1 - 2k)c_0 + 2k \sum_{j=1}^{k-1} \frac{c_j}{j} + (k + 1)c_k \right)
\]

\[
= c_0 [a_{n,0} - a_{n,1} - 3a_{n,2} - 5a_{n,3} - \cdots - (2n - 1)a_{n,n}]
\]

\[
+ c_1 [2a_{n,1} + 2(2a_{n,2} + 3a_{n,3} + \cdots + (n - 1)a_{n,n-1} + na_{n,n})]
\]

\[
+ c_2 [3a_{n,2} + (3a_{n,3} + 4a_{n,4} + \cdots + (n - 1)a_{n,n-1} + na_{n,n})]
\]

\[
+ c_3 [4a_{n,3} + \frac{2}{3} (4a_{n,4} + 5a_{n,5} + \cdots + (n - 1)a_{n,n-1} + na_{n,n})]
\]

\[
+ \cdots
\]

\[
+ c_i \left[ (i + 1)a_{n,i} + \frac{2}{i} ((i + 1)a_{n,i+1} + (i + 2)a_{n,i+2} + \cdots + (n - 1)a_{n,n-1} + na_{n,n}) \right]
\]

\[
+ \cdots
\]
\[ + c_{n-1} \left[ na_{n,n-1} + \frac{2}{n-1} (na_{n,n}) \right] + c_n [(n + 1)a_{n,n}] \]
\[ = c_0 [(a_{n,0} + a_{n,1} + \cdots + a_{n,n}) - 2(a_{n,1} + 2a_{n,2} + 3a_{n,3} + \cdots + (n - 1)a_{n,n-1} + na_{n,n})] \]
\[ + c_1 [2(a_{n,1} + 2a_{n,2} + 3a_{n,3} + \cdots + (n - 1)a_{n,n-1} + na_{n,n})] + \sum_{k=2}^n (k + 1)c_k a_{n,k} \]
\[ + 2 \sum_{k=2}^{n-1} c_k [(a_{n,k+1} + a_{n,k+2} + \cdots + a_{n,n}) + \frac{1}{k} (a_{n,k+1} + 2a_{n,k+2} + \cdots + (n - k)a_{n,n})]. \]

Now, using (2.3) and observing that \( a_{n,1} = 2 \sum_{k=1}^n ka_{n,k} \) and \( a_{n,k} = 0 \) for \( k > n \), we can write
\[(Ax)_n = c_0 a_{n,0}^{(1)} + (c_1 - c_0) a_{n,1}^{(*)} + \sum_{k=2}^n c_k \left[ (k + 1)a_{n,k} + 2a_{n,k+1} + \left( \frac{2}{k} \right) a_{n,k+1}^{(2)} \right] \]
\[ = c_0 a_{n,0}^{(1)} + (c_1 - c_0) a_{n,1}^{(*)} + \sum_{k=2}^n c_k a_{n,k}^{(*)}. \]

By the linearity of the operators \( \mathcal{M} \) and \( S \), we have
\[ S(\mathcal{M}((Ax)_n)) \]
\[ = c_0 S(\mathcal{M}(a_{n,0}^{(1)})) + (c_1 - c_0) S(\mathcal{M}(a_{n,1}^{(*)})) + \sum_{k=2}^n c_k S(\mathcal{M}(a_{n,k}^{(*)})) \]  \( \geq 0, \) \( \text{ (3.1)} \)

by conditions (i), (ii) and (iii). Thus, the sequence \( \{(Ax)_n\} \) is mean-starshaped.

Conversely, assume that the matrix \( A \) preserves mean-starshape of sequences. Suppose that condition (ii) does not hold. Therefore, in the first column of the matrix \( A^{(*)} \), there exists an \( N \geq 2 \) such that \( S(\mathcal{M}(a_{N,1}^{(*)})) = \alpha \neq 0 \). Choose a sequence \( \{x_n\} \) such that \( x_n = -2n \alpha \). Therefore, from (2.2), we get that
\[ c_0 = x_0 = 0, \quad c_1 = \frac{1}{2} (x_1 + x_0) = -\alpha \]

and for \( k \geq 2, \)
\[ c_k = \frac{1}{k+1} x_0 - \frac{2}{k^2 - 1} \sum_{j=1}^{k-1} x_j + \frac{1}{k+1} x_k \]
\[ S(\mathcal{M}(Ax)_N) = c_1 S(\mathcal{M}(a_{N,1}^*)) + \sum_{k=2}^{N} c_k S(\mathcal{M}(a_{N,k}^*)) \]
\[ = (\alpha)(\beta) = -\beta^2, \]

which contradicts that the transformed sequence \((Ax)_n\) is mean-starshaped. Hence, condition (ii) must be true.

Next, suppose that condition (i) does not hold. That is, there exists an \(N \geq 2\), such that
\[ S(\mathcal{M}(a_{N,0}^*)) = \beta \neq 0. \]
Choose a mean-starshaped sequence \(\{x_n\}\) such that \(x_n = (2n-1)\beta\). Therefore, from
\[ (2.2) \quad c_0 = x_0 = -\beta, \quad c_1 = \frac{1}{2}(x_0 + x_1) = 0, \]
and for \(k \geq 2\)
\[ c_k = \frac{1}{k+1}x_0 - \frac{2}{k^2-1} \sum_{j=1}^{k-1} x_j + \frac{1}{k+1}x_k \]
\[ = -\frac{\beta}{k+1} - \frac{2}{k^2-1} \sum_{j=1}^{k-1} (2j-1)\beta + \frac{1}{k+1}(2k-1)\beta \]
\[ = \frac{\beta}{k+1}(2k-2) - \frac{2\beta}{k^2-1}(1 + 3 + \ldots + (2k-3)) \]
\[ = 0. \]

Thus, from (3.1) and noting that \(S(\mathcal{M}(a_{n,1}^*)) = 0\) for each \(n \geq 2\),
\[ S(\mathcal{M}(Ax)_N) = c_0 S(\mathcal{M}(a_{N,0}^*)) \]
\[ = (\beta)(\beta) = -\beta^2, \]

which contradicts that the transformed sequence \((Ax)_n\) is mean-starshaped. Therefore, condition (i) must be true.

We will now show that condition (iii) is necessary for the matrix \(A\) to preserve the mean-starshape of the sequences. Since we have established conditions (i) and (ii), the equation (3.1) simplifies to
\[ S(\mathcal{M}((Ax)_n)) = \sum_{k=2}^{n} c_k S(\mathcal{M}(a_{n,k}^*)) \quad (3.2) \]
We need to prove that for each fixed $k \geq 2$,

$$S(\mathcal{M}(a_{n,k}^{(e)})) \geq 0 \text{ for } n \geq k.$$  

Suppose the above condition is not true. Therefore, for some $k \geq 2$, the $k$-th column of the matrix $A^{(e)}$ is not mean-starshaped. Then, for that $k$, there exists an $N \geq k$ such that

$$S(\mathcal{M}(a_{N,k}^{(e)})) = \lambda < 0.$$  

Now, we construct a sequence $\{x_n\}$ as follows.

$$x_n = \begin{cases} 0, & \text{if } n < N, \\ -(N+1)\lambda, & \text{if } n = N, \\ -2\lambda \frac{n}{N}, & \text{if } n > N. \end{cases}$$

Since the sequence $\{x_n\}$ can be represented in the form of (2.1) the corresponding values of $c_j's$ can be calculated using (2.2). By routine algebraic calculation, we find that

$$c_0 = c_1 = \cdots = c_{N-1} = 0,$$

$$c_N = -\lambda,$$

$$c_{N+1} = c_{N+2} = \cdots = 0.$$  

Therefore, the sequence $\{x_n\}$ is mean-starshaped. Thus, (3.2) reduces to

$$S(\mathcal{M}(Ax)_n) = c_N S(\mathcal{M}(a_{N,k}^{(e)})) = (-\lambda)(\lambda) = -\lambda^2 < 0,$$

which is a contradiction. Hence, the theorem.

Next, we observe that the three sufficient conditions in Theorem 3.1 for a matrix to preserve mean-starshape of the sequences can be replaced by weaker conditions. This will make it easier to check, if any matrix $A$ preserves mean-starshape of the sequences.

**Theorem 3.2.** A lower triangular matrix $A$ preserves mean-starshape of sequences, if for each $n = 2, 3, \cdots$,

(i) $S(a_{n,0}^{(1)}) = 0$  

(ii) $S(a_{n,1}^{(e)}) = 0$  

(iii) $S(a_{n,k}^{(e)}) \geq 0$ for $k \geq 2$. 


Proof. Using Lemma 2.5, we see that conditions (i) and (ii) are equivalent to

\[ S(\mathcal{M}(a_{n,0}^{(1)})) = 0 \quad \text{and} \quad S(\mathcal{M}(a_{n,1}^{(*)})) = 0. \]

Now, if \( \{x_n\} \) is a mean-starshaped sequence, then the corresponding \( c_k \)'s are non-negative for \( k \geq 2 \). Using Lemma 1.4 in equation (3.1), we have that

\[ S(\mathcal{M}(Ax)_n) = \sum_{k=2}^{n} c_k S(\mathcal{M}(a_{n,k}^{(*)})) \geq 0. \]

4. Examples. The lower triangular matrix of ones defined by \( A = [a_{n,k}] \) where

\[ a_{n,k} = \begin{cases} 1, & \text{if } k \leq n, \\ 0, & \text{if } k > n \end{cases} \]

does not preserve the mean-starshape of sequences.

From the Definition 2.2, we have

\[ a_{n,1}^{(*)} = 2 \sum_{k=1}^{n} k a_{n,k} \]
\[ = 2(1 + 2 + 3 + \cdots + n) \]
\[ = n(n + 1). \]

Now, for \( n \geq 2 \),

\[ S(\mathcal{M}(a_{n,1}^{(*)})) = \frac{\mathcal{M}(a_{n,1}^{(*)}) - \mathcal{M}(a_{n-1,1}^{(*)})}{n - 1} \]
\[ = \frac{1}{n} \left[ a_{1,1}^{(*)} + \cdots + a_{n,1}^{(*)} \right] - \frac{1}{n-1} \left[ a_{1,1}^{(*)} + \cdots + a_{n-1,1}^{(*)} \right] \]
\[ = \frac{1}{n(n+1)} [1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1)] \]
\[ - \frac{1}{n(n-1)} [1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n] \]
\[ = \frac{n+2}{3} - \frac{n+1}{3} \neq 0. \]

Thus, condition (ii) of Theorem 3.1 fails, implying that \( A \) does not preserve the mean-starshape of sequences.

Next, we consider one of the classical matrices. Cesaro matrix \( C = [C_{n,k}] \) is given by [3, page 44]

\[ C_{n,k} = \begin{cases} \frac{1}{n+1}, & \text{for } k \leq n, \\ 0, & \text{for } k > n. \end{cases} \]
It is easy to see that the corresponding matrices $C^{(1)}$ and $C^{(2)}$ are given by

\[
C^{(1)}_{n,k} = \begin{cases} 
\frac{n-k+1}{n+1}, & \text{if } k \leq n, \\
0, & \text{if } k > n,
\end{cases}
\]

\[
C^{(2)}_{n,k} = \begin{cases} 
\frac{(n-k+1)(n-k+2)}{2(n+1)}, & \text{if } k \leq n, \\
0, & \text{if } k > n.
\end{cases}
\]

We will show that the three conditions of Theorem 3.2 are true in the case of Cesaro matrix. First, for $n \geq 2$, 

\[
S(C^{(1)}_{n,0}) = S(1) = 0.
\]

Thus, condition (i) of Theorem 3.2 is satisfied. Next, we see that 

\[
C^{(\star)}_{n,1} = 2 \sum_{k=1}^{n} k C_{n,k} = n.
\]

So, for $n \geq 2$, 

\[
S(C^{(\star)}_{n,1}) = S(n) = 0.
\]

Now, to check condition (iii) of Theorem 3.2, we observe from (2.3) that 

\[
C^{(\star)}_{n,k} = (k+1)C_{n,k} + 2C^{(1)}_{n,k+1} + \frac{2}{k}C^{(2)}_{n,k+1}
\]

\[
= \frac{k+1}{n+1} + 2\frac{(n-k)}{n+1} + \frac{(n-k)((n-k+1))}{k(n+1)}
\]

\[
= \frac{1}{k(n+1)} \left[ k(k+1) + 2k(n-k) + (n-k)(n-k+1) \right]
\]

\[
= \frac{n}{k}.
\]

For a fixed $k \geq 2$, 

\[
S(C^{\star}_{n,k}) = \frac{n}{nk} - \frac{n-1}{(n-1)k} = 0.
\]

So, by Theorem 3.2 the Cesaro matrix preserves mean-starshape of sequences.

Another well-known lower triangular matrix is Nörlund matrix $N_p[n,k]$ which is given by [3, page 43]

\[
N_{n,k} = \begin{cases} 
p_{n-k}, & \text{if } k \leq n, \\
0, & \text{if } k > n
\end{cases}
\]

for $p_0 > 0$ and $P_n = \sum_{k=0}^{n} p_k$. It is easy to see that

\[
p_k + 2p_{k-1} + 3p_{k-2} + \cdots + (k+1)p_0 = P_k + P_{k-1} + \cdots + P_0.
\]
Further more, we see that

\[ N_{n,k}^{(1)} = \begin{cases} 
\frac{P_n - k}{P_n}, & \text{if } k \leq n, \\
0, & \text{if } k > n 
\end{cases} \]

and

\[ N_{n,k}^{(2)} = \begin{cases} 
\frac{1}{P_n} (p_n - k + 2p_{n-k-1} + \cdots + (n-k+1)p_0), & \text{if } k \leq n, \\
0, & \text{if } k > n. 
\end{cases} \]

In order to determine if Nörlund matrix preserves mean-starshape of sequences, first we consider condition (i) of Theorem 3.1 which is equivalent to \( S(N_{n,0}^{(1)}) = 0 \) for \( n \geq 2 \) by Lemma 1.5. Since \( N_{n,0}^{(1)} = 1 \) for all \( n \), \( S(N_{n,0}^{(1)}) = 0 \). Thus, the condition (i) of Theorem 3.1 holds for any Nörlund matrix. Next, we consider condition (ii) of Theorem 3.1, which is equivalent to \( S(N_{n,1}^{(s)}) = 0 \) for \( n \geq 2 \).

For each \( n \geq 2 \), we need the following lemmas.

**Lemma 4.1.** If the Nörlund matrix satisfies condition (ii) of Theorem 3.1, then we get the following equivalent statements. For \( n \geq 2 \),

(i) \( \frac{P_{n-1} + \cdots + P_0}{nP_n} = \frac{P_{n-2} + \cdots + P_0}{(n-1)P_{n-1}} \)

(ii) \( \frac{P_{n-1} + \cdots + P_0}{nP_n} = \frac{P_0}{P_1} \)

(iii) \( \frac{P_0}{P_1} = \frac{2P_0}{P_2} = \frac{3P_0}{P_3} = \cdots = \frac{nP_0}{P_n} \)

(iv) \( p_n = \frac{1}{n!} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)p_0 \) and \( P_n = \frac{1}{n!} \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n)p_0 \)

where \( \alpha = \frac{P_1}{P_0} \).

The proof is obvious.
Lemma 4.2. If the Nörlund matrix satisfies condition (ii) of Theorem 3.1, then for each \( k \geq 2 \) and \( n \geq k \),

\[
N_{n,k}^{(*)} = \frac{(n-k+1)(n-k+2)\cdots n}{(\alpha+n-k)(\alpha+n-k+1)\cdots(\alpha+n)} \left[ (k+1) \left( \alpha + \frac{2(n-k)}{k} \right) + \frac{2(n-k)(n-k-1)}{k(\alpha+1)} \right].
\]

Proof. Using (2.4) we can write

\[
N_{n,k}^{(*)} = (k+1)N_{n,k} + \frac{2}{k} \sum_{j=k+1}^{n} jN_{n,j}.
\]

Using (4.1), the above equation can be written as

\[
N_{n,k}^{(*)} = (k+1)\frac{p_{n-k}}{P_n} + \frac{2}{k} \sum_{j=k+1}^{n} j\frac{p_{n-j}}{P_n}
\]

\[
= \frac{1}{P_n} \left[ (k+1)p_{n-k} + \frac{2(k+1)}{k} \sum_{j=k+1}^{n} p_{n-j} + \frac{2}{k} \left( p_{n-k-2} + 2p_{n-k-3} + \cdots + (n-k-1)p_0 \right) \right].
\]

Using (4.2) in the last sum, we get

\[
N_{n,k}^{(*)} = \frac{1}{P_n} \left[ (k+1)p_{n-k} + \frac{2}{k} (k+1)p_{n-k-1} + \frac{2}{k} (P_{n-k-2} + \cdots + P_0) \right].
\]

Using (ii) of Lemma 4.1, we get

\[
N_{n,k}^{(*)} = \frac{1}{P_n} \left[ (k+1)p_{n-k} + \frac{2}{k} (k+1)P_{n-k-1} + \frac{2}{k} \left( P_{n-k-2} + \cdots + P_0 \right) \right].
\]

Using (iv) of Lemma 4.1 and simplifying we get the required expression for \( N_{n,k}^{(*)} \).

Lemma 4.3. If the Nörlund matrix satisfies condition (ii) of Theorem 3.1, then for each \( k \geq 2 \) and \( n \geq k \),

\[
nN_{n+1,k}^{(*)} - (n+1)N_{n,k}^{(*)} = \frac{(n+1)!((k+1)(k-1)\alpha(\alpha-1)}{(n-k+1)!((\alpha+n-k)\cdots(\alpha+n+1))}.
\]

Proof. In Lemma 4.2, replacing \( n \) by \( n+1 \) in the expression for \( N_{n,k}^{(*)} \) we obtain the expression for \( N_{n+1,k}^{(*)} \). Now, we can compute the difference

\[
nN_{n+1,k}^{(*)} - (n+1)N_{n,k}^{(*)}
= \frac{(n+1)n\cdots(n-k+2)}{(\alpha+n-k)\cdots(\alpha+n+1)}
\]
\[
\times \left[ n(\alpha + n - k) \left( (k + 1)\alpha + \frac{2(k + 1)(n - k + 1)}{k} + \frac{2(n - k + 1)(n - k)}{k(\alpha + 1)} \right) \\
-(n - k + 1)(\alpha + n + 1) \left( (k + 1)\alpha + \frac{2(k + 1)(n - k)}{k} + \frac{2(n - k)(n - k - 1)}{k(\alpha + 1)} \right) \right]
\]

\[
= \frac{(n + 1)n \cdots (n - k + 2)}{(\alpha + n - k) \cdots (\alpha + n + 1)} (k + 1) \left[ \alpha^2(k - 1) - \alpha(k - 1) \right].
\]

Multiplying both the numerator and the denominator of the above expression by \((n - k + 1)!\) we obtain

\[
nN_{n+1,k}^{(\ast)} - (n + 1)N_{n,k}^{(\ast)} = \frac{(n + 1)!(k + 1)(k - 1)\alpha(\alpha - 1)}{(n - k + 1)!(\alpha + n - k) \cdots (\alpha + n + 1)}.
\]

**Lemma 4.4.** If the Nörlund matrix satisfies condition (ii) of Theorem 3.1, then for each \(k \geq 2\) and \(n \geq k\),

\[
S(\mathcal{M}(N_{n,k}^{(\ast)})_n) = \frac{(n - 2)!(k - 1)\alpha}{(n - k)!(\alpha + n - k)(\alpha + n - k + 1) \cdots (\alpha + n)},
\]

(4.4)

where \(\alpha = \frac{p_1}{p_0}\).

**Proof.** We prove the lemma by induction. First, we notice that in the \(k\)-th column of the matrix \([N_{n,k}^{(\ast)}]\), the entries are zero if \(n < k\). For \(n = k\), from (2.6) and (4.1), we see that

\[
\mathcal{M}(N_{n,k}^{(\ast)}) = N_{k,k} = \frac{p_0}{P_k}.
\]

Now, substituting \(n = k\), the left side of (4.4) becomes

\[
S(\mathcal{M}(N_{k,k}^{(\ast)})) = S(N_{k,k}) = \frac{p_0}{kP_k}
\]

and the right side of (4.4) reduces to

\[
\frac{(k - 1)!}{(\alpha + 1) \cdots (\alpha + k)} = \frac{p_0}{kP_k}
\]
from (iv) of Lemma 4.1.

Next, for a fixed \( k \), replacing \( n \) by \( k + 1 \) in (2.7), we obtain

\[
S(\mathcal{M}(N_{k+1,k}^{(s)})) = \frac{1}{k(k+1)(k+2)} \sum_{m=k}^{k+1} \left[ (m-1)N_{m,k}^{(s)} - mN_{m-1,k}^{(s)} \right]
\]

\[
= \frac{1}{k(k+1)(k+2)} \left[ (k-1)N_{k+1,k}^{(s)} - kN_{k+1,k}^{(s)} + kN_{k-1,k}^{(s)} - (k+1)N_{k,k}^{(s)} \right].
\]

Combining similar terms and noting that \( N_{k-1,k}^{(s)} = 0 \), we have

\[
S(\mathcal{M}(N_{k+1,k}^{(s)})) = \frac{1}{k(k+1)(k+2)} \left[ -2N_{k,k}^{(s)} + kN_{k+1,k}^{(s)} \right].
\]

From (2.4) and from the definition (4.1) of Norlund matrix, we obtain

\[
S(\mathcal{M}(N_{k+1,k}^{(s)})) = \frac{1}{k(k+1)(k+2)} \left[ -2(k+1)N_{k,k}^{(s)} + \frac{k(k+1)p_0}{P_k} + \frac{k(k+1)p_1}{P_{k+1}} + \frac{2(k+1)p_0}{P_{k+1}} \right].
\]

Using (iv) of Lemma 4.1, the above equation simplifies to

\[
S(\mathcal{M}(N_{k+1,k}^{(s)})) = \frac{(k-1)!(k-1)\alpha}{(\alpha+1)\cdots(\alpha+k+1)}.
\]

Thus, we have shown that (4.4) is true for \( n = k \) and \( n = k + 1 \). Now, assume that (4.4) is valid for a natural \( n \geq k + 1 \). It is sufficient to show that

\[
S(\mathcal{M}(N_{n+1,k}^{(s)})) = \frac{(n-1)!(k-1)\alpha}{(n-k+1)!(\alpha+n-k+1)\cdots(\alpha+n+1)}.
\]

From our assumption and noting that \( N_{0,k}^{(s)} = 0 \) we have from (2.7) that

\[
\sum_{m=k}^{n} \left[ (m-1)N_{m,k}^{(s)} - mN_{m-1,k}^{(s)} \right] = \frac{(n+1)!(k-1)\alpha}{(n-k)!(\alpha+n-k)\cdots(\alpha+n)}.
\]

(4.5)

First, we write using (2.7)

\[
S(\mathcal{M}(N_{n+1,k}^{(s)})) = \frac{1}{n(n+1)(n+2)} \sum_{m=k}^{n+1} \left[ (m-1)N_{m,k}^{(s)} - mN_{m-1,k}^{(s)} \right]
\]

\[
= \frac{1}{n(n+1)(n+2)} \left[ \sum_{m=k}^{n} \left( (m-1)N_{m,k}^{(s)} - mN_{m-1,k}^{(s)} \right) + \left( nN_{n+1,k}^{(s)} - (n+1)N_{n,k}^{(s)} \right) \right].
\]
Then, using (4.5) in the first term and using Lemma 4.3 in the last term on the right side of the above equation, we get

\[
S(\mathcal{M}(N_{n+1,k}^{(\star)})) = \frac{1}{n(n+1)(n+2)} \left[ \frac{(n+1)!(k-1)\alpha}{(n-k)!(\alpha+n-k) \cdots (\alpha+n)} \right. \\
+ \left. \frac{(n+1)!(k+1)(k-1)\alpha(\alpha-1)}{(n-k+1)!(\alpha+n-k) \cdots (\alpha+n+1)} \right].
\]

Simplifying the right side, we obtain

\[
S(\mathcal{M}(N_{n+1,k}^{(\star)})) = \frac{(n-1)!(k-1)\alpha}{(n-k+1)!(\alpha+n-k+1) \cdots (\alpha+n+1)}.
\]

Hence, the lemma.

Thus, \( S(\mathcal{M}(N_{n,k}^{(\star)})) \geq 0 \) for each \( k \geq 2 \) and \( n \geq k \) which yields condition (iii) of Theorem 3.1. Thus, we get the following theorem.

**Theorem 4.5.** The Nörlund matrix \( N_{p,n,k} \) preserves the mean-starshape of sequences if and only if the sequence \( \{p_n\} \) satisfies the condition that

\[
p_n = \frac{1}{n!} \alpha(\alpha+1) \cdots (\alpha+n-1) p_0
\]

for some \( \alpha > 0 \).

Next, we discuss another familiar matrix, Euler matrix, which is given by [3, page 54]

\[
E_r[n,k] = E_{n,k} = \begin{cases} 
\binom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \leq n, \\
0, & \text{if } k > n 
\end{cases} \tag{4.6}
\]

where \( 0 < r < 1 \). The corresponding matrices \( E^{(1)}_{n,k} \), \( E^{(2)}_{n,k} \) and \( E^{(\star)}_{n,k} \) are given by

\[
E^{(1)}_{n,k} = \begin{cases} 
\sum_{j=k}^{n} \binom{n}{j} r^j (1-r)^{n-j}, & \text{if } k \leq n, \\
0, & \text{if } k > n 
\end{cases}
\]

\[
E^{(2)}_{n,k} = \begin{cases} 
\sum_{j=k}^{n} (j-k+1) \binom{n}{j} r^j (1-r)^{n-j}, & \text{if } k \leq n, \\
0, & \text{if } k > n 
\end{cases}
\]

and

\[
E^{(\star)}_{n,k} = (k+1)E_{n,k} + \frac{2}{k} \sum_{j=k+1}^{n} jE_{n,j}. \tag{4.7}
\]
We will prove that all three conditions of Theorem 3.1 are true in the case of Euler matrix. Since,

\[ E_{n,0}^{(1)} = (r + 1 - r)^n = 1, \]

we have \( S(E_{n,0}^{(1)}) = 0 \), which is equivalent to \( S(M(E_{n,0}^{(1)})) = 0 \).

Since \( E_{n,1}^{(*)} = 2(E_{n,1} + 2E_{n,2} + 3E_{n,3} + \cdots + nE_{n,n}) \), we have

\[
S(E_{n,1}^{(*)}) = \frac{1}{n} E_{n,1}^{(*)} - \frac{1}{n-1} E_{n-1,1}^{(*)} \]

\[
= \frac{2}{n} \left( \sum_{k=1}^{n} kE_{n,k} \right) - \frac{2}{n-1} \left( \sum_{k=1}^{n-1} kE_{n-1,k} \right) \]

\[
= \frac{2(1-r)^n}{n} \sum_{k=1}^{n} k \left( \frac{n}{k} \right) \left( \frac{r}{1-r} \right)^k - \frac{2(1-r)^{n-1}}{n-1} \sum_{k=1}^{n-1} k \left( \frac{n-1}{k} \right) \left( \frac{r}{1-r} \right)^k \]

Letting \( x = \frac{r}{1-r} \) and using the operator \( x \frac{d}{dx} \), we can write that

\[
S(E_{n,1}^{(*)}) = \frac{2(1-r)^n}{n} \left( x \frac{d}{dx} \right) \left( \sum_{k=1}^{n} \left( \frac{n}{k} \right) x^k \right) - \frac{2(1-r)^{n-1}}{n-1} \left( x \frac{d}{dx} \right) \left( \sum_{k=1}^{n-1} \left( \frac{n-1}{k} \right) x^k \right) \]

\[
= \frac{2(1-r)^n}{n} \left( x \frac{d}{dx} \right) [(1 + x)^n - 1] - \frac{2(1-r)^{n-1}}{n-1} \left( x \frac{d}{dx} \right) [(1 + x)^{n-1} - 1] \]

\[
= \frac{2(1-r)^n}{n} x[n(1 + x)^{n-1}] - \frac{2(1-r)^{n-1}}{n-1} x[(n-1)(1 + x)^{n-2}] \]

\[
= 2r - 2r = 0, \]

which is equivalent to \( S(M(E_{n,1}^{(*)})) = 0 \). Thus, conditions (i) and (ii) of Theorem 3.1 hold for Euler matrix.

Next, in order to prove that \( S(M(E_{n,k}^{(*)})) > 0 \), we need the following lemmas.

**Lemma 4.5.** The matrix \( E^{(*)} \) satisfies the following equation. For each \( k \geq 2 \) and \( n \geq k \)

\[
nE_{n+1,k}^{(*)} - (n+1)E_{n,k}^{(*)} = \binom{n+1}{k} r^k (1-r)^{n-k} (k-1)[k+1-(n+2)r]. \]

**Proof.** Using (4.7) and (4.6), we see, after some algebraic manipulation that

\[
nE_{n+1,k}^{(*)} - (n+1)E_{n,k}^{(*)} \]

\[
= r^k (1-r)^{n-k} \left[ (k^2 - 1) \binom{n+1}{k} (1-r) - (k+1) \binom{n+1}{k} (n-k+1) r \right] \]

\[
+ 2r(k+1) \binom{n+1}{k+1} \]

\[
+ \frac{2}{k} \sum_{j=1}^{n-k} \frac{r^{j+1}}{(1-r)^2} (j+k) \left\{ \binom{n+1}{j+k+1} (j+k) - \binom{n+1}{j+k} (n+1-j-k) \right\} \].

Since the quantity inside the braces is zero and noting that \( \binom{n+1}{k+1}(k+1) = \binom{n+1}{k}(n+1-k) \), the above equation simplifies to

\[
ne_{n+1,k} - (n+1)e_{n,k} = r^k(1-r)^{n-k} \left[ (k^2 - 1) \binom{n+1}{k} (1-r) - r \binom{n+1}{k} (n+1-k)(k-1) \right]
\]

\[
= r^k(1-r)^{n-k} \binom{n+1}{k} (k-1) [k + 1 - (n+2)r].
\]

Hence, the lemma.

**Lemma 4.6.** In the matrix \([E_{n,k}^{(*)}]\), for each \( k \geq 2 \) and \( n \geq k \), we have

\[
S(\mathcal{M}(E_{n,k}^{(*)})) = \frac{r^k(1-r)^{n-k}(k-1)(n-2)!}{k!(n-k)!}. \tag{4.8}
\]

**Proof.** We prove the lemma by induction. First, we notice that in the \( k \)-th column of the matrix \([E_{n,k}^{(*)}]\), the entries are zero if \( n < k \).

For \( n = k \), from (2.6) and (4.6), we see that

\[
\mathcal{M}(E_{k,k}^{(*)}) = E_{k,k} = r^k.
\]

Now, substituting \( n = k \), the left side of (4.8) becomes

\[
S(\mathcal{M}(E_{k,k}^{(*)})) = S(r^k) = \frac{r^k}{k!}
\]

and the right side of (4.8) reduces to

\[
\frac{r^k(k-1)(k-2)!}{k!} = \frac{r^k}{k!}.
\]

Next, for a fixed \( k \), replacing \( n \) by \( k+1 \) in (2.7), we obtain

\[
S(\mathcal{M}(E_{k+1,k}^{(*)})) = \frac{1}{k(k+1)(k+2)} \sum_{m=k}^{k+1} \left[ (m-1)E_{m,k}^{(*)} - mE_{m-1,k}^{(*)} \right]
\]

\[
= \frac{1}{k(k+1)(k+2)} \left[ -2E_{k,k}^{(*)} + kE_{k+1,k}^{(*)} \right].
\]

From (2.4), we obtain

\[
S(\mathcal{M}(E_{k+1,k}^{(*)})) = \frac{r^k(1-r)(k-1)}{k!} = r^k(1-r)(k-1) \frac{(k-1)!}{k!}.
\]
Thus, we have shown that (4.8) is true for \( n = k \) and \( n = k + 1 \). Now, assume that (4.8) is valid for a natural \( n \geq k + 1 \). It is sufficient to show that

\[
S(M(E_{n+1,k}^{(s)}))_n = \frac{r^k(1-r)^{n-k+1}(k-1)(n-1)!}{k!(n-k+1)!}.
\]

From our assumption, we have from (2.7) that

\[
\sum_{m=k}^{n} [(m-1)E_{m,k}^{(s)} - mE_{m-1,k}^{(s)}] = \frac{r^k(1-r)^{n-k}(k-1)(n-2)!n(n^2-1)}{k!(n-k)!}.
\]  

(4.9)

First, we write using (2.7)

\[
S(M(E_{n+1,k}^{(s)}))_n = \frac{1}{n(n+1)(n+2)} \left[ \sum_{m=k}^{n} [(m-1)E_{m,k}^{(s)} - mE_{m-1,k}^{(s)}] + nE_{n+1,k}^{(s)} - (n+1)E_{n,k}^{(s)} \right].
\]

Then, we use Lemma 4.5 and the equation (4.9) to obtain that

\[
S(M(E_{n+1,k}^{(s)}))_n = \frac{1}{n(n+1)(n+2)} \left[ \frac{r^k(1-r)^{n-k}(k-1)(n-2)!n(n^2-1)}{k!(n-k)!} 
\right. \\
+ \left. \left( \frac{n+1}{k} \right) r^k(1-r)^{n-k}(k-1)(k+1-(n+2)r) \right] \\
= \frac{r^k(1-r)^{n-k+1}(k-1)(n-1)!}{k!(n-k+1)!}.
\]

Hence, the result. Thus, the Euler matrix preserves the mean-starshape of a sequence.

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