Dulac Functions for Transformed Vector Fields

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Abstract

In this paper we study the existence of Dulac functions for planar differential systems with perturbations or under algebraic operations (addition and multiplication) on vector fields, and also, the transformation of vector fields under affine transformations. We give some applications and examples in order to illustrate the applicability of the results.

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1 Introduction

There are some criteria that allow us to rule out the existence of periodic orbits in the plane; between them, we will take particular interest in studying the
Bendixson Dulac criterion. It is well known that Bendixson-Dulac criterion is a very useful tool for investigation of limit cycles (see [2], [3], [5], [6]). For convenience, we recall the Bendixson-Dulac criterion, see ([4], p. 137).

Theorem 1.1 (Bendixson-Dulac criterion) Let $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ and $h(x_1, x_2)$ be functions $C^1$ in a simply connected domain $\Omega \subset \mathbb{R}^2$ such that

$$\frac{\partial (f_1 h)}{\partial x_1} + \frac{\partial (f_2 h)}{\partial x_2}$$

does not change sign in $\Omega$ and vanishes at most on a set of measure zero. Then the system

\[
\begin{cases}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2), \\
(x_1, x_2) &\in \Omega,
\end{cases}
\]  

(1)

does not have periodic orbits in $\Omega$.

A function as in the theorem is called a Dulac function, despite the relevance of Bendixson-Dulac’s criterion it suffers the drawback that there is no algorithm for finding $h$ functions. In this letter, we study the existence of Dulac functions for planar vector fields with perturbations or under algebraic operations on vector fields, the operations are addition and multiplication. Finally, we prove the existence of Dulac functions for planar differential systems under affine change of variables.

2 Results and Discussion

Consider the vector field $F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, then the system (1) can be rewritten in the form

\[
\dot{x} = F(x), \quad x = (x_1, x_2) \in \Omega.
\]  

(2)

As usual the divergence of $F$ is defined by $\text{div}(F) = \text{div}(f_1, f_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$.

We consider $C^0(\Omega, \mathbb{R})$ the set of continuous functions and define

$$\mathcal{F}^\pm_\Omega := \{ f \in C^0(\Omega, \mathbb{R}^\pm \cup \{0\}) : \text{vanishes only on a measure zero set} \},$$

and $\mathcal{F}_\Omega := \mathcal{F}^-_\Omega \cup \mathcal{F}^+_\Omega$. Also for the simply connected region $\Omega$, we introduce

$$\mathcal{D}^\pm_\Omega(F) := \{ h \in C^1(\Omega, \mathbb{R}) : k := \frac{\partial (hf_1)}{\partial x_1} + \frac{\partial (hf_2)}{\partial x_2} \geq 0, \ k \in \mathcal{F}^\pm_\Omega \}.$$

Thus a Dulac function of (1) is an element in $\mathcal{D}_\Omega(F) := \mathcal{D}^+_\Omega(F) \cup \mathcal{D}^-_\Omega(F)$. 

2.1 Bendixson-Dulac theorem and perturbations

A problem in constructing Dulac functions is the difficulty in making a priori estimates, so our first step is to introduce a kind of compactness.

Recall that a (nonempty) open set $\Omega_0$ is compactly contained in an open set $\Omega \subseteq \mathbb{R}^2$, denoted $\Omega_0 \subset\subset \Omega$, if $\overline{\Omega_0} \subset \Omega$ and $\overline{\Omega_0}$ is compact. Note that if $\Omega_0 \subset\subset \Omega$, then there is an open set $V$ such that $\Omega_0 \subset\subset V \subset\subset \Omega$.

The following result weakens the conditions of Bendixson-Dulac criterion

**Proposition 2.1** Let $\Omega$ be a simply connected open set and $F \in C^1(\Omega, \mathbb{R}^2)$ be a vector field. If for every simply connected open set $\Omega_0 \subset\subset \Omega$ we have $\mathcal{D}_{\Omega_0}(F) \neq \emptyset$, then the system (2) does not have periodic orbits in $\Omega$.

**Proof:** We proceed by contradiction suppose that the system admits a periodic solution $\gamma$ in $\Omega$. Denote by $\text{int}(\gamma)$ the open region bounded by $\gamma$, note that $\text{int}(\gamma) \subset\subset \Omega$, then there is an open set (simply connected) $U$ such that $\text{int}(\gamma) \subset\subset U \subset\subset \Omega$, therefore we have $\mathcal{D}_U(F) \neq \emptyset$ which contradicts that such periodic orbits exist.

Now we present a result on Dulac functions for systems with perturbations

**Proposition 2.2** Let $\Omega$ be a simply connected open set. Let $F, G : \Omega \to \mathbb{R}^2$, be $C^1$ vector fields, suppose that $F$ admits a Dulac function $h$ such that $\text{div}(hF) \neq 0$, then given a simply connected open set $\Omega_0 \subset\subset \Omega$ exists $\epsilon_0 > 0$ such that $\mathcal{D}_{\Omega_0}(F + \epsilon G) \neq \emptyset$, $\forall \epsilon$ with $\epsilon_0 > \epsilon > 0$.

**Proof:** For convenience suppose $\text{div}(hF) > 0$. Given a simply connected open set $\Omega_0 \subset\subset \Omega$, consider an open set (compactly contained) $V$ with $\Omega_0 \subset V \subset \overline{V} \subset \Omega$. Taking $m_0, m_1 > 0$ such that $\text{div}(hF) > m_0$ and $|\text{div}(hG)| \leq m_1$ on $\overline{V}$. From the linearity of $\text{div}(\cdot)$ we have

$$\text{div}(h(F + \epsilon G)) = \text{div}(hF + \epsilon hG) = \text{div}(hF) + \epsilon \text{div}(hG),$$

thus for $\epsilon_0 > 0$ small enough we have $\text{div}(h(F + \epsilon G)) > 0$ for $\epsilon_0 > \epsilon > 0$, i.e.,

$$\mathcal{D}_{\Omega_0}(F + \epsilon G) \neq \emptyset.$$ 

Which leads the desired result. $\square$

**Example 2.3** Consider the Lotka-Volterra equations

$$\begin{align*}
\dot{x}_1 &= x_1(r_1 - k_1x_1 - b_{12}x_2) = f_1(x_1, x_2), \\
\dot{x}_2 &= x_2(r_2 - k_2x_2 - b_{21}x_1) = f_2(x_1, x_2),
\end{align*}$$
since these equations model biological systems in which two species interact, we consider \( x_1 > 0 \) and \( x_2 > 0 \). Moreover take \( k_1k_2 \geq 0 \) with \( k_1, k_2 \) are not both zero. Denote by \( F = (f_1, f_2) \) the vector field associated to the equation, for \( h(x_1, x_2) = (x_1x_2)^{-1} \), we get
\[
\text{div}(hF) = -\frac{k_1x_1 + k_2x_2}{x_1x_2} < 0,
\]
thus for any compact region \( R_0 \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \) and \( G \) vector field there is \( \epsilon_0 > 0 \) such that if \( 0 < \epsilon < \epsilon_0 \) then \( F + \epsilon G \) does not support limit cycles on \( R_0 \).

### 2.2 Sum of vector fields and multiplication by scalar functions

Now we study the problem of determining which vector fields we can add to one given for the resulting system admits a Dulac function

**Proposition 2.4** Assume that \( h \in \mathcal{D}_\Omega^+(F) \), then for every \( C^1 \) vector field \( G \) with \( (\pm)\text{div}(hG) \geq 0 \) we have \( h \in \mathcal{D}_\Omega(F + G) \).

**Proof:** Suppose for convenience that \( h \in \mathcal{D}_\Omega^+(F) \) and let \( G \) such that \( \text{div}(hG) \geq 0 \) hence by linearity \( \text{div}(h(F + G)) \in \mathcal{F}_\Omega^+ \), thus \( h \in \mathcal{D}_\Omega^+(F + G) \). \( \square \)

**Example 2.5** Consider the Lotka-Volterra equations of example 2.3. We incorporate a constant immigration term, \( e_1 \) and \( e_2 \), to the two populations, respectively, and we subject each population to their own proportional harvesting effort, \( h_1 \) and \( h_2 \) respectively (i.e., the selective harvesting), the equation become
\[
\begin{align*}
\dot{x}_1 &= x_1(r_1 - k_1x_1 - b_{12}x_2) + e_1 - h_1x_1 = f_1(x_1, x_2) + g_1(x_1, x_2), \\
\dot{x}_2 &= x_2(r_2 - k_2x_2 - b_{21}x_1) + e_2 - h_2x_2 = f_2(x_1, x_2) + g_2(x_1, x_2).
\end{align*}
\]
Doing \( G = (g_1, g_2) \), with \( h(x_1, x_2) = \frac{1}{x_1x_2} \) we have
\[
\text{div}(hG) = -\frac{e_1x_2 + e_2x_1}{x_1^2x_2^2} < 0,
\]
Thus \( \mathcal{D}_\Omega(F + G) \neq \emptyset \), for \( \Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \).

Recall that a function \( f \in C^1(\Omega, \mathbb{R}) \) is an integrating factor of the vector field \( X \) if \( \text{div}(fX) = 0 \).

**Corollary 2.6** Suppose \( h \in \mathcal{D}_\Omega(F) \) and \( h \) is an integrating factor of \( G \) then \( \mathcal{D}_\Omega(F + G) \neq \emptyset \).
Example 2.7  Let $h$ be a Dulac function for $F$ on $\Omega$ and $X_h = (-\frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial x_1})$ the Hamiltonian field associated to $h$, since $h$ is an integrating factor of $X_h$, by above corollary we get $\mathcal{D}_\Omega(F + X_h) \neq \emptyset$.

The next analyzes the existence of Dulac functions of vector fields obtained by multiplying a vector field by a real function

**Proposition 2.8**  Let $\Omega$ be a simply connected open set and $g : \Omega \to \mathbb{R}$ be a $C^1$ function. Suppose there is $h \in \mathcal{D}_\Omega(F)$ then the following are satisfied

i) If $\frac{h}{g} \in C^1(\Omega, \mathbb{R})$, then $\mathcal{D}_\Omega(gF) \neq \emptyset$.

ii) If $\frac{f_i}{g} \in C^1(\Omega, \mathbb{R})$, $i = 1, 2$, then $\mathcal{D}_\Omega(\frac{1}{g}F) \neq \emptyset$.

**Proof:**  Case i). Since $h$ is a Dulac function for $F$, then $\text{div}(hF) \in \mathcal{F}_\Omega$. Therefore $\text{div}((\frac{h}{g})(gF)) \in \mathcal{F}_\Omega$, that is $\mathcal{D}_\Omega(gF) \neq \emptyset$. Case ii). Similar arguments to the previous case prove that $hg \in \mathcal{D}_\Omega(\frac{1}{g}F)$. The proof is complete. \qed

A consequence of Proposition 2.8 is given in the following result useful in determining when a system does not support Dulac functions.

**Corollary 2.9**  Let $\Omega \subset \mathbb{R}^2$ be a simply connected set, $g : \Omega \to \mathbb{R}$ be a $C^1$ function and $G := gF = gf_1\frac{\partial}{\partial x_1} + gf_2\frac{\partial}{\partial x_2}$ be a $C^1$ vector field. If $\mathcal{D}_\Omega(G) \neq \emptyset \Rightarrow \mathcal{D}_\Omega(F) \neq \emptyset$.

### 2.3 Dulac Functions and changing variables

A classic method for studying differential equations are changing variables, we recall this result, suppose a $C^1$ differential equation $\dot{x} = f(x)$, $x \in \Omega$ an open set. If $T$ is a diffeomorphism from $\Omega$ to $\Sigma$ then the differential equation (or the vector field) can be pushed forward as $\dot{y} = g(y)$ on $\Sigma$ and the well known expression

$$\dot{y} = DT \left(T^{-1}(y)\right) f(T^{-1}(y)), \ y \in \Sigma,$$

is the new differential equation on $\Sigma$. A natural question is whether to make a change of coordinates preserving certain amounts, in particular if a Dulac function becomes a function of Dulac after a change of variables. The following is a direct calculation

**Proposition 2.10**  We consider the system (1) and a change of variable $T : \Omega \to \Sigma$ with inverse $S(y_1, y_2) = (S_1(y_1, y_2), S_2(y_1, y_2)) = (x_1, x_2)$, denote by $\dot{y} = g(y) = (g_1(y_1, y_2), g_2(y_1, y_2))$ the transformed equation. Let

$$J(y_1, y_2) := \left(\frac{\partial S_1}{\partial y_1} \frac{\partial S_2}{\partial y_2} - \frac{\partial S_1}{\partial y_2} \frac{\partial S_2}{\partial y_1}\right) (y_1, y_2)$$
be the Jacobian of \( S \). Given \( \overline{h} : \Sigma \to \mathbb{R} \) a \( C^1 \)-function and \( h(x_1, x_2) := \overline{h}(y_1, y_2) \), then
\[
\left( \frac{\partial \overline{h} g_1}{\partial y_1} + \frac{\partial \overline{h} g_2}{\partial y_2} \right)(y_1, y_2) = \left( \frac{\partial h f_1}{\partial x_1} + \frac{\partial h f_2}{\partial x_2} \right)(S_1(y_1, y_2), S_2(y_1, y_2)) - \frac{\overline{h}}{J} \left( \frac{\partial J}{\partial y_1} g_1 + \frac{\partial J}{\partial y_2} g_2 \right)(y_1, y_2)
\]

**Corollary 2.11** If we transform a system by an affine change of variables, then the original system supports a Dulac function if and only if the transformed system has a Dulac function.

**Proof:** For an affine transformation \( S \) we have \( J = 1 \), thus \( \frac{\partial J}{\partial y_1} = \frac{\partial J}{\partial y_2} = 0 \). The proof is complete. \( \square \)

**Example 2.12** Consider the system
\[
\begin{align*}
\dot{x}_1 &= -5x_1^2 - 6x_1 x_2 - 2x_2^2 + x_2, \\
\dot{x}_2 &= 10x_1^2 + 12x_1 x_2 + 4x_2^2 + x_1 - x_2,
\end{align*}
\]
applying the affine change of variable \((x_1, x_2) \rightarrow (y_1 = 2x_1 + x_2 + 1, y_2 = x_1 + x_2)\), system becomes
\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -y_1 - y_2 + y_1^2 + y_2^2,
\end{align*}
\]
Since \( \overline{h}(y_1, y_2) = e^{-2y_1} \) is a Dulac function for this system then the first system has to \( h(x_1, x_2) = e^{(-4x_1 - 2x_2 - 2)} \) as Dulac function by corollary 2.11.

**Example 2.13** We consider the classic SIS epidemiological model with disease-induced death
\[
\begin{align*}
\dot{x}_1 &= \lambda - \beta x_1 x_2 - \mu x_1 + \delta x_2, \\
\dot{x}_2 &= \beta x_1 x_2 - (\alpha + \mu + \delta) x_2, \\
\end{align*}
\]
(3)
The structure of above system suggests the change of variables \((x_1, x_2) \rightarrow (y_1 = x_1 + x_2, y_2 = x_2)\), system becomes
\[
\begin{align*}
\dot{y}_1 &= \lambda - \mu y_1 - \alpha y_2, \\
\dot{y}_2 &= \beta(y_1 - y_2)y_2 - (\alpha + \mu + \delta)y_2,
\end{align*}
\]
(4)it is clear that \( \overline{h}(y_1, y_2) = (y_2)^{-1} \) is a Dulac function of
\[
\begin{align*}
\dot{y}_1 &= \lambda - \mu y_1, \\
\dot{y}_2 &= \beta(y_1 - y_2)y_2,
\end{align*}
\]
and \( \text{div}(\overline{h}(-\alpha y_2, -(\alpha + \mu + \delta)y_2)) = 0 \), thus by corollary 2.6 \( \overline{h}(y_1, y_2) = (y_2)^{-1} \) is a Dulac function of (4), therefore by corollary 2.11 \( h(x_1, x_2) = (x_2)^{-1} \) is the corresponding Dulac function of (3).


dulac functions

References


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