Some New Modular Identities of Ramanujan Continued Fraction

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Abstract

In this paper, we establish some new modular identities of Ramanujan continued fraction.

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1 Introduction

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

\[ f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |ab| < 1, \]  

\[ = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty} \]
where \((a; q)_0 = 1\) and \((a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots \).

Following Ramanujan, we defined

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q)_\infty}{(q; q)_\infty}, \tag{1.2}
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.3}
\]

\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty \tag{1.4}
\]

and

\[
\chi(q) := (-q; q^2)_\infty. \tag{1.5}
\]

Ramanujan recorded many \(q\)-continued fractions and some of their explicit values in his second notebook [8] and in his lost notebook [9]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [1, p. 11, Entry 11]:

\[
\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a - b}{1 - q} + \frac{(a - bq)(aq - b)}{1 - q^3} + \frac{q(a - bq^2)(aq^2 - b)}{1 - q^5} + \cdots \tag{1.6}
\]

where either \(q\), \(a\), and \(b\) are complex numbers with mod \(q < 1\), or \(q\), \(a\), and \(b\) are complex numbers with \(a = bq^m\) for some integer \(m\). Several elegant \(q\)-continued fractions can be expressed in terms of Ramanujan’s theta-functions. The most famous of them is the celebrated Rogers-Ramanujan continued fraction \(R(q)\) is defined as

\[
R(q) := \frac{q^{1/5}f(-q, -q^4)}{f(-q^2, -q^3)} = q^{1/5} + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}, \quad |q| < 1, \tag{1.7}
\]

On page 365 of his Lost Notebook [9], Ramanujan recorded five identities showing the relationships between \(R(q)\) and five continued fractions \(R(-q), R(q^2), R(q^3), R(q^4)\), and \(R(q^5)\). He also recorded these identities at the scattered places of his Notebooks [8]. L. J. Rogers [10] established the modular equations relating \(R(q)\) and \(R(q^n)\) for \(n=2, 3, 5, \) and \(11\). The last of these equations cannot be found in Ramanujan’s works.

The Ramanujan’s cubic continued fraction \(G(q)\) is defined as

\[
G(q) := \frac{q^{1/3}f(-q, -q^5)}{f(-q^3, -q^4)} = q^{1/3} + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \cdots}}, \quad |q| < 1, \tag{1.8}
\]

The continued fraction (1.8) was first introduced by Ramanujan in his second letter to G. H. Hardy [6]. He also recorded the continued fraction
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(1.8) on page 365 of his Lost Notebook [9] and claimed that there are many results for \( G(q) \) similar the results obtained for the famous Rogers-Ramanujan continued fraction (1.7).

The Ramanujan Göllnitz-Gordon continued fraction [6, p. 44], [4], [9] is defined as follows:

\[
L(q) := \frac{q^{1/2} f(-q^3, -q^5)}{f(-q, -q^7)} = \frac{q^{1/2}}{1 + \frac{q^2}{1 + q^3 + \frac{q^4}{1 + q^5 + \cdots}}, \ |q| < 1, \tag{1.9}
\]

Motivated by the above cited works on the continued fractions, in this paper, we established the modular relation between continued fraction \( H(q) \) and \( H(q^n) \).

\[
H(q) := \frac{1 - \varphi(-q)}{1 + \varphi(-q)} = \frac{q}{1 - q + \frac{q^3}{1 - q^3 + \frac{q^5}{1 - q^5 + \cdots}}}. \tag{1.10}
\]

The continued fraction (1.10) was first established by Nipen Saikia [7], they established some modular relations connecting \( H(q) \) and \( H(q^n) \) and some explicit evaluations of \( H(q) \).

2 Preliminary results

In this section, we collect the necessary results required to prove our main results.

**Lemma 2.1.**

\[
\varphi(q) = \sqrt{z} \tag{2.1}
\]

and

\[
\varphi(-q) = \sqrt{z(1 - t)^{1/4}}. \tag{2.2}
\]

For the proofs of (2.1) and (2.2), see [2, Entry 10(i),(ii), p.122].

**Lemma 2.2.** If \( \beta \) is of degree 2 over \( \alpha \), then

\[
(1 - \sqrt{1 - \alpha})(1 - \sqrt{\beta}) = 2\sqrt{\beta(1 - \alpha)}. \tag{2.3}
\]

For a proof of (2.3), see [1, Entry 17.3.1, p.385].

**Lemma 2.3.** If \( \beta \) has degree 3 over \( \alpha \), then

\[
(\alpha \beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1. \tag{2.4}
\]

For a proof of (2.4), see [2, Entry 5(ii), p.230].

**Lemma 2.4.** If \( \beta \) has degree 4 over \( \alpha \), then

\[
(1 - \sqrt{1 - \alpha})(1 - \sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1 - \alpha)}. \tag{2.5}
\]
For a proof of (2.5), see [1, Entry 17.3.2, p.385].

Lemma 2.5.

\[ \varphi(-q) = \frac{1 - H(q)}{1 + H(q)} \]  

(2.6)

For the proofs of (2.6), easily from (1.10).

3 Relation Between \( H(q) \) and \( H(q^n) \)

Theorem 3.1. If \( u := H(q), \ v := H(-q), \ z := H(q^2), \) and \( w := H(-q^2), \)

then

\[
(-w^2 + 2wv^2 - w^2z^2 - 2w^2zv^2 - z^2 - 2zv^2 + 4wz - 1 + 2wz^2v^2)w^2
+ (2w^2v + 2v - 4wz^2v + 2w^2z^2v + 4zv + 4w^2zv - 4wv - 8wzv + 2z^2v^2)u
+ 2wz^2 - w^2z^2v^2 - 2wz^2w + 2w - w^2v^2 - v^2 - z^2v^2 - 2z + 4wzv^2 = 0.
\]

(3.1)

Proof. From the equations (2.1) and (2.2), we get

\[
\frac{\varphi(-q)}{\varphi(q)} = (1-t)^{1/4}, \quad 0 < t < 1.
\]

(3.2)

The equation (2.3) can be written as

\[
\beta = \left[ \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}} \right]^2.
\]

(3.3)

Employing the equations (2.6) and (3.2) in the above equation (3.3), we obtain (3.1).

Theorem 3.2. If \( u := H(q), \ v := H(-q), \ z := H(q^3), \) and \( w := H(-q^3), \)

then

\[
12w^3z^2v + 6w^2z^4v^2 - 12w^2zv - 12w^3zv^2 + 24w^2z^2v^2 + 12w^3z^2v^3
+ 6w^4z^2v^2 - 12w^3z^3v^2 - 12w^2zv^3 - 12w^2z^2v - 12w^3v^2 + w^4zv^4
- 4w^3z^3v + 4wz^4v^3 + 6w^2z^2v^4 + 12w^2z^3v^4 - 4w^3z^4v^4 - 12w^3z^2v^4 + 4w^4z^4v
- 12w^2z^3v^3 - 4w^4z^3v + 12w^2z^2v + (12w^3z^2v + 4w^3z^4v^3 + 6w^2z^4v^2 - 12w^2zv
- 4w^4zv - 12w^3z^2v^2 + 24w^2z^3v^2 + 12w^3z^2v^3 + 4wz^4v - 4w^3zv^4 + 6w^4z^2v^2 + 1
- 12w^3z^4v^2 - 12w^2zv^3 - 12w^2zv^4 + 12w^2z^2v + 12w^2z^2v^4 + 6w^2z^3v^2 - 12wz^3v^2
- 4w^4z^3v^3 - 12w^2z^3v^4 + 12w^2z^2v^4 + 6w^2z^2v^4 - 4w^3z^3v + 6w^2z^2v - 4wz
+ 6w^2z^2v - 4w^3v - 4z^3v + 6w^2v^2 - 4zv^3 + 4wz^3 + 4w^4v + z^4v^4)u^4 + (12w^4z^3v^2
+ 40w^3z^3v + 72w^2z^2v^2 + 40wzv + 12w^4z^2v^2 + 24w^3zv - 12w^2z^2v - 12w^2z^4v
+ 12w^3zv^2 + 24wzv^3 - 12w^3z^4v^2 + 40w^3z^3v - 72w^2z^2v^2 - 4w^4z^4v - 12w^4z^2v^2
\]

\]
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Proof.

\[
-72w^3z^2v^2 + 24w^3z^3v^3 + 72w^2z^3v^2 + 4w^4z^3v^4 + 24wz^3v + 12w^2z^3v^4 \\
-12w^4z^3v^3 - 4w^4z^3v^4 - 72w^2z^3v^2 - 12w^3z^2v + 12w^4z^3v^4 - 12w^4z^2v \\
+4w^3zv^2 - 72w^2z^3v - 4v - 12wz^2 - 4wz^4 + 12w^2z + 12w^2z^3 - 12w^3z^2 \\
+4w^4z - 4w^3 + 4z^2 - 12z^2v - 12z^3v - 12w^2v - 12w^3v^3 + 12zv^2 + 4zv^4 \\
-12wv^2 - 4wv^4 - 12w^3z^3v^2 - 4w^4z^3v^4 + 6z^3v^2 - 4z^3v^4v^3 + (72w^3z^2v \\
+12w^3z^3v^2 + 24w^2z^4v^2 - 72w^2zv - 12w^4zv + 6w^2z^4v^4 - 72w^3zv^2 \\
+168w^2z^2v^2 + 72w^3z^3v^3 + 12w^4v - 12w^3zv^4 + 24w^4z^2v^2 - 72w^3z^3v^2 \\
+6w^4z^2v^4 - 72w^2zv^3 - 72wz^2v^2 - 72w^2z^3v - 12wzv^4 - 12w^4z^3v + 12w^4z^3v^3 \\
+6w^4z^4v^2 + 24w^2z^2v^4 + 72wz^2v^3 - 12w^3z^3v^4 - 72wz^3v^2 - 12w^4z^3v^3 \\
+12w^3z^3v - 72w^2z^3v^3 + 6z^2 - 12w^4z^3v^3 + 72w^2zv - 12w^3zv^4 - 12w^3v \\
+24w^3z^2 + 6w^2z^3 - 12w^2z^3 + 6w^4z^2 - 12wz + 24^2z^2 - 12zv \\
+6z^2v^4 + 12wv + 12w^3v - 12z^3v + 24^2z^2v + 6w^2v^4 - 12zv^3 + 12w^3v \\
+12w^3v^3 + 6w^4v^2 - 12z^3v^3 + 6z^2v^2 + 6w^2v^2 + 6w^2v^2 + (12w^4z^2v^2 - 4z^4v \\
+72w^2zv^2 + 24wzv + 12w^4zv^3 + 40w^3zv - 12w^2z^4v - 12w^2z^4v^3 + 12w^2zv^4 \\
+40wzv^3 - 12w^3z^4v^2 - 4w^4z^4v^3 + 24w^3z^3v - 72w^2z^2v^2 - 12w^3z^4v \\
-72w^4z^2v^2 + 12w^3z^3v^3 + 72w^2z^3v^2 + 40wz^3v + 12w^2z^3v^4 + 4w^4z^4v^4 \\
-12w^3z^2v^3 - 72w^3z^3v^3 - 4w^3v^2 - 12w^3z^2v^4 - 12w^2z^2v^4 - 12w^3z^4v^3 \\
+24w^3z^3v^3 - 72w^2z^2v^2 + 4z - 4w - 12w^2z + 12w^2z^3 - 12w^2z^2 - 4w^3 \\
-4w^3z^3 + 4w^3z^2 - 12z^2v - 12z^2v^3 + 12w^2zv - 4w^4v + 24wz^3v^3 - 12w^2v^3 \\
+12zv^2 - 12w^2v^2 - 12w^2v^2 - 12w^3v^4 + 12z^3v^2 + 4z^3v^4)u - 4wz^3 + 6w^2z^2 \\
-4w^3z + w^4 + z^4 + v^4 + 6z^2v^2 - 4zv + 4wv + 6w^2v^2 + 4w^3v^3 - 4z^3v^3 = 0.
(3.4)

\[
\text{Proof.}\quad \text{The equation (2.3) can be written as}
\]

\[
\alpha\beta = [1 - \{(1 - \alpha)(1 - \beta)\}]^{1/4}.
(3.5)
\]

Employing the equations (2.6) and (3.2) in the above equation (3.5), we obtain (3.4).

\[\square\]

Theorem 3.3. If \(u := H(q), v := H(-q), z := H(q^4),\) and \(w := H(-q^4),\) then

\[
-6v^4w^4z^2 - 6v^4w^2z^4 + 16v^4z^3v - 36v^4w^2z^2 - 24v^4wz^2 - v^4w^4z^4 \\
+16v^4w^3z^3 + 4v^4w^4z^3 + 24v^4w^2z^2 - 4v^4w^3z^4 + 4v^4w^4z + 24v^4w^2z^3 \\
+(-1 + 48v^4wz^2 - 8v^4w^4z^3 - 48v^4w^2z + 8v^4w^3z^4 - 8v^4w^3z - 48v^4w^2z^3
\]
\[ + 48v^4w^3z^2 + 8v^4wz + 16wz + 4z - 4w - 6w^2 - 6z^2 - 24wz^2 + 16wz^3 \\
- 4wz + 24w^2z - 36wz^2 + 24w^2z^3 - 6w^2z^4 + 16wz^3 - 24wz^3 - 8v^4z \\
+ 16w^3z^3 - 4w^3z^4 + 4w^4z - 6w^4z^2 + 4w^4z^3 - w^4z^4 - 4w^3 - w^4 + 4z^3 \\
- z^4 + 8v^4w^3 - 8v^4w^3 + 8v^4w^4 - 24v^4w^3z^2 + 16v^4wz - 4v^4wz^4 \\
+ (144v^2w^3z^2 - 36v^2w^3z^2 + 24v^2w^3z^4 - 144v^2w^2z + 144v^2w^2z + 96v^2wz \\
- 36v^2w^2z^4 + 96v^2w^3z^3 + 96v^2wz^3 - 6w^2w^4z^4 - 144v^2w^2z^4 - 24v^2w^4z \\
- 24v^2w^4z^3 + 24v^2w^4z^4 - 216v^2w^2z^2 + 24v^2w^3z^2 - 6w^2 - 24v^2 - 36v^2w^2 \\
+ 24v^2w^3 - 6w^2w^4 - 36v^2z^2 - 24v^2z^2 - 6v^2z^4 + 24v^2w^2w^2 + (16v^3w^4 \\
+ 16v^3w^3z^4 - 32vw^4z^3 + 32vw^4z^2 + 24v^3w^4z^2 + 96v^3w^3z^2 - 96v^3w^3z^3 \\
+ 32vw^4z^3 - 192vw^3z^2 + 144v^3w^3z^2 + 192vw^2z + 96v^3w^3z^2 \\
- 64v^3w^3z^2 - 64v^3w^3z^3 - 192vw^3z^2 - 32vw^4z^4 + 4v^3w^3z^4 - 64v^3wz \\
- 64v^3w^3z^2 - 96v^3w^3z^3 - 64v^3w^3z^4 - 64v^3w^3z^5 + 24v^3w^3z^4 - 32vw^3 + 32vz \\
+ 32vz^3 - 32vw + 4v^3 - 16v^3 + 16wv^3 + 24v^2v^3 + 16v^3w^3 + 4v^3w^3 + 4v^3w^4 + 4v^3z^4 \\
- 16v^3z^3 + 24v^2z^4)u + 16^4w^3z^3 + (-32v^3w^4z^4 - 32v^3w^3z^4 + 16v^4z^4 \\
- 16vw^4z + 24vw^2z^4 - 192v^3w^3z^2 + 192v^3w^2z^3 + 16vw^4z^3 + 144v^2w^2z^2 \\
- 64vw^3z^3 + 96vw^2z - 64vw^2z^2 - 64vw^2z - 96vw^2z^2 - 192vw^3z^2 \\
+ 32v^3w^4z^3 + 96vw^3z^2 + 16vw^3z^4 + 24vw^4z^2 + 32vw^4z + 192vw^3z^2 \\
- 64vw^3z^2 + 4vw^4z^4 + 4v + 24vw^2 + 16wv^3 + 4vw^4 - 16vz + 24wz^2 - 16vz^3 \\
+ 4w^4z + 16vw + 16vz^3 - 32vw^3 - 32vw^3z^2 + 32vw^3z^3)u - 8z + 8w + 48w^2z^2 \\
- 48w^2z - 48w^2z^2 + 48w^2z^3 + 8w^2z^4 - 8w^2z^4 - 8w^4z - 8w^4z^3 + 8w^3 - 8z^3 + 8w^4z^3 \\
- v^4 - 6v^4w^2 - 4v^4w^2 - v^4w^4 + 4v^4z - 6v^4z^2 + 4v^4z^3 - v^4z^4 - 4v^4w = 0. \tag{3.6} \\
\]

Proof. The equation (2.4) can be written as

\[ \beta = \left[ \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}} \right]^4. \tag{3.7} \]

Employing the equations (2.6) and (3.2) in the above equation (3.7), we obtain (3.6). \hfill \square

References


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