Abstract. This paper is concerned with the asymptotic theory of shallow rivulets. The asymptotic solution based on a series expansion, by a simple variant of lubrication theory, whose validity depends on the smallness of the filament’s height to width ratio.

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1. Introduction

A rivulet is a stream of liquid, of a finite cross-sectional extent, flowing down a solid surface. The cross-sectional domain $D$ of the rivulet is bounded partly by a solid boundary $B$, and partly by the liquid surface $S$. Rivulets play an important role in hydrodynamics. The relevance of such flows to a number of industrial processes has been noted by Towell & Rothfeld [12] (packed contractors, tricklephase vectors, condensation and imprison on surface). Allen & Biggen [1] (heat exchange, gas absorption), and Young & Davis [15] (melting and casting of metals). The efficiency of most of these processes is strongly dependent on the cross-sectional shape of the rivulet, and its quality, being greatest when the rivulet is stably maintained in the form of a shallow film of large width. Investigation of the mechanical principles which determine this shape, and its quality, is the main objective of theoretical work. Lanbein [6] and Roy & Schwartz [9] studied the stability of rivulets using variational methods and by exploring whether or not it is energetically favourable for a rivulet to split up into several subrivulets. Schmuki & Laso [10], Myers & liang [7],

An important aspect of filament’s flow is normally the very large ratio of surface area to cross-sectional area. The greater this parameter the more effective in the fluid filament’s in processes of heat exchange or gas absorption.

Problems concerning the draining of viscous films down inclined surfaces have received much attention in the literature. Duffy & Moffatt [4] use a thin-film approximation to study the steady spreading or contraction of viscous liquid supplied (at a prescribed rate) on a near vertical plane. Benilov [2] examined the linear stability of a capillary rivulet under the assumption that it is shallow enough to be described by the lubrication approximation. And considered the case where the liquid-substrate interaction involves a finite-width hysteresis interval. Rivulets are assumed to be sufficiently shallow and, thus, satisfy the lubrication approximation.

This paper is restricted to the asymptotic theory of strictly steady shallow rivulets of uniform liquid down a vertical plane. It is therefore not concerned with the analysis of stability. Throughout, the effect of the ambient atmosphere is neglected.

2. General Theory

In the restricted class, the domain $D$ is necessarily uniform of the form shown in fig. 1, where an appropriate Cartesian coordinates system is also indicated. Although a steady rivulet is an elementary example of unidirectional viscous flow, many aspect of which were treated in the nineteenth century. The theory of the nondegenerate cases in which $L$ is bounded ($L$ is the half-width of the rivulet) appears first to have been discussed by Towell & Rothfeld [12]. Since the assumed conditions require the liquid pressure to be everywhere uniform. Non-zero values of the surface of the rivulet is a segment of circular cylinder. The shape of the domain $D$ is therefore uniquely determined by the single parameter,

$\frac{h}{L} = \frac{H}{L} = \frac{1 - \cos \alpha}{\sin \alpha}$

where $H$ is the maximum thickness of the rivulet, and $\alpha$ is the contact angle. A sketch of the inverse relation $\alpha(h)$ is included in fig. 1.

3. Formulation and Scaling of the Problem

The flow is assumed to be steady and uniform. The direction of the flow is parallel to negative $Z$-axis. Thus, if $U, V$ and $-W$ are the velocity components in the flow then, $U = V = 0$, and

$W = W(X, Y)$ with a further assumption that the liquid is incompressible viscous, the Navier-Stokes equations for the laminar flow reduce to
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\[ \frac{\partial P}{\partial X} = \frac{\partial P}{\partial Y} = 0 \]

\[ \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial P}{\partial Z} = -\frac{g}{\nu} \]

where \( P \) is pressure, \( g \) is gravity, and \( \nu \) is kinematics viscosity.

We now assume that the ambient gas has no significant effect. The boundary conditions appropriate to the liquid gas interface \( S \) thus, become:

(i) - The shear stress condition

\[ \left( \frac{\partial W}{\partial n} \right)_s = 0 \]

where \( n \) is the coordinate normal to the free surface.

(ii) - The normal stress condition from the Laplace’s condition

\[ (\Delta P)_s = \frac{\sigma}{R}, \]

where \( R \) is the local radius of curvature of \( S \), and \( \sigma \) is the surface tension of the liquid.

(iii) - The no-slip condition of the solid boundary \( B \)

\[ W(X, Y) = 0. \]

The scales of the velocity, and length are most naturally fixed by the volume flux \( Q \), which is defined by
\[
Q = \int_{D} W(X, Y) dX dY
\]

Since the ambient gas has no mechanical effect, and the free surface is independent of \( Z \) then, from equation (3.4), it follows that the pressure \( P_s \) is independent of \( Z \) therefore,

\[
\frac{\partial P_s}{\partial Z} = 0.
\]

On the other hand from eq.(3.1) we get that \( \frac{\partial P}{\partial Z} \) is a function of \( Z \), only then from eq(3.1) and (3.7) we get

\[
\frac{\partial P}{\partial Z} = 0
\]

everywhere across the rivulet, so that eq(3.2) becomes:

\[
\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{g}{v}.
\]

In the scaling of the problem we have:

\[
W(X, Y) = \lambda^2 w(x, y) \left( Q \cdot \frac{g}{v} \right)^{\frac{1}{2}}
\]

\[
L = \lambda \left( Q \cdot \frac{v}{g} \right)^{\frac{1}{4}}
\]

\[
x = \frac{X}{L} , \ y = \frac{Y}{L}.
\]

Where \( \lambda \) is the main non-dimensional parameter to be determined as a function of \( h \). Consistency of the transformation then, demands

\[
\lambda^4 \int_{D} w(x, y) dxdy = 1.
\]

With eqs (3.10), (3.11), and (3.12) the Navier - Stokes equation of motion (3.9) becomes

\[
\nabla^2 w(x, y) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1.
\]

The boundary conditions (3.3), and (3.5) will be
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(3.15) \[ w(x, y) = 0, \text{ at } y = 0, \]

(3.16) \[ \left( \frac{\partial w}{\partial n} \right)_s = 0. \]

The unique solution to this Poissons problem then yields the function \( w(x, y) \), and hence the parameter \( \lambda \), for each value of \( h \).

4. Asymptotic Theory for shallow Rivulet

Analytical results have been obtained only for small values of \( h \). The limiting solution as \( h \rightarrow 0 \) (interpreted as \( L \rightarrow \infty \) for a fixed value of \( H \) ) is the one case that belongs to the early period of viscous flow theory: it being a special case of Kelvin’s [4] solution for two-dimensional flow down an inclined plane. Adaptation of Kelvin’s solution to \( \alpha \) uniformly valid approximation in the scaled coordinates, namely

(4.1) \[ w(x, y) = \frac{1}{2} \left( 2h \left( 1 - x^2 \right) - y \right), \quad 0 < y < h \left( 1 - x^2 \right), \]

which may be regarded as a variant of the lubrication theory initiated by Reynolds [8] was included in the results of Towell & Rothfeld [12].

A method of developing higher order lubrication theory in this context was outlined by Allen & Biggen [1]. In terms of the resealed variables

(4.2) \[ \xi = x, \eta = \frac{y}{h}, \Phi = \frac{w(x, y)}{h^2}, \]

the governing equation (3.14) becomes

(4.3) \[ \frac{\partial^2 \Phi}{\partial \eta^2} = -1 - h^2 \frac{\partial^2 \Phi}{\partial \xi^2}, \]

and the equation of the liquid surface \( s \) becomes

(4.4) \[ \eta_s = \frac{-(1 - h^2) + \left\{ (1 - h^2)^2 + 4h^2 \left( 1 - \xi^2 \right) \right\}^{\frac{1}{2}}}{2h^2}, \]

on which the boundary condition will be

(4.5) \[ \left( \frac{\partial \Phi}{\partial \eta} \right)_s = -1 - h^2 \frac{d\eta_s}{d\xi} \left( \frac{\partial \Phi}{\partial \xi} \right)_s. \]

for the formal expansions
\begin{align}
\eta_s &= f_0(\xi) + h^2 f_2(\xi) + h^4 f_4(\xi) + \cdots \tag{4.6} \\
\Phi &= \Phi_0(\xi, \eta) + h^2 \Phi_2(\xi, \eta) + h^4 \Phi_4(\xi, \eta) + \cdots \tag{4.7} \\
\text{the solutions are:} \\
f_0(\xi) &= 1 - \xi^2, \tag{4.8} \\
f_2(\xi) &= \xi^2 \left(1 - \xi^2\right), \tag{4.9} \\
f_4(\xi) &= -\xi^2 \left(1 - \xi^2\right) \left(1 - 2\xi^2\right), \tag{4.10} \\
\Phi_0(\xi, \eta) &= \eta \left(1 - \xi^2\right) - \frac{1}{2} \eta^2, \tag{4.11} \\
\Phi_2(\xi, \eta) &= -\eta \left(1 - \xi^2\right) \left(1 - 6\xi^2\right) + \frac{1}{3} \eta^3, \tag{4.12} \\
\Phi_4(\xi, \eta) &= \eta \left(1 - \xi^2\right) \left(7 - 78\xi^2 + 100\xi^4\right) - \frac{1}{3} \eta^3 \left(7 - 36\xi^2\right). \tag{4.13}
\end{align}

The leading terms \(f_0\) and \(\Phi_0\) where used by Towell & Rothfeld \[12\]; the second order terms \(f_2\) and \(\Phi_2\) were calculated by Young & Davis \[15\] eq.(4.9a,b); and the third order terms \(f_4\) and \(\Phi_4\) have been calculated by author. It appears from the discussion by Young & Davis \[15\] that these regular expansions are uniformly valid throughout the entire domain \(D\), including the contact lines, to all orders in \(h\). Thus, in the usual terminology of expansion theory the ‘outer’ expansion represents the complete solution to the problem; a phenomena which Young & Davis \[15\] note is common in lubrication theory.

The solution for \(\lambda\) is obtained from (3.13)

\begin{equation}
\lambda^{-4} = \left(\frac{32}{105}\right) h^3 \left(1 - \frac{1}{9} h^2 + \frac{142}{99} h^4 + \cdots\right), \tag{4.14}
\end{equation}

the leading term of which was calculated by Towell & Rothfeld \[12\]. (But eq.(25) in that paper should read \(f(\theta) = -\frac{160}{34}\) as \(\theta \to 0\); and higher order approximation in \(\theta\) are not consistent with first order lubrication theory. This correction increases their value of \(\lambda\) by 10\%, so that the asymptotes at small values of \(\Omega\) in fig.5 of the same paper should be raised by this amount.)

The first, second and third order approximations to \(\lambda(h)\) are shown in fig. 2. The inverse relation \(\alpha(h)\) of eq.(2.1) is
Using (4.14) the distribution of velocity in the free surface is

\[ w_s = \left( \frac{6}{35} \right)^{\frac{1}{2}} h^{\frac{5}{2}} \left[ 1 - \frac{16}{45} h^2 + \frac{306913}{178200} + \cdots \right]. \]

The successive approximation by equation (4.16) are shown in fig.3.

5. Observed Data

Only a very few observations have been made of vertical rivulets. Towell & Rothfeld [15] reported observations of water rivulet on glass plate inclined at 10° to the vertical, for which values \( Q \) less than 0.10ml/sec gave stable flows. In this range, they found that the measured values for the width were usually about 30% less than the theoretical values according to the leading term of eq.(4.14). Since the correction for the inclination of the boundary to the vertical is less than 2% in this case, the discrepancy may be partly due to errors of order of 0.5° in the measured of small contact angles, which were here as small as 3°. But this alone would scarcely account for the whole of the error.
The only other relevant observations are those of water rivulets on an almost vertical Mylar plate reported by Culkin & Davis [3]. These water- Mylar systems exhibited strong contact angle hysteresis, with receding and advancing contact angles of $30^\circ \pm 3^\circ$ and $77^\circ \pm 3^\circ$, respectively. The steady-state solution was therefore not unique, and it became necessary to define a precise process of rivulet formation in order to achieve a unique steady state. When the volume flux $Q$ was increased slowly and steadily from zero, Culkin & Davis[3] observed that stable vertical rivulets occurred for $Q < 1.0 \text{ml/sec}$, but a meandering instability set in for higher values of $Q$.

6. Conclusion

The successive approximations represented by eqs.(4.14) and (4.16) are shown in fig-2- and fig-3- respectively. A purely internal comparison of the curves suggests that the second approximation of lubrication theory provides a reasonable approximation to both $\lambda$ and $w_s$ for values of $h$ up to about 0.5, with a rapidly increasing error as $h$ increases beyond that values.

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References


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