Fixed Point Theorems with Implicit Function in Metric Spaces

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Abstract
In this paper, weakly compatible and occasionally weakly compatible mappings are discussed in detailed and generalized common fixed point theorems of two and four mappings are proved in metric space with implicit function. Our results extends and generalists many well known results.

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1 Introduction
In recent years, Popa [7] used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its
unifying power as an implicit function can cover several contraction conditions at the same time which includes known as well as unknown contraction conditions. This fact is evident from examples furnished in Popa [7]. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. In this paper we have discussed the dislocated topologies associated with a metric space and also proved a common fixed point theorem of Presic type which extends and generalises the well known Banach contraction principle and known results.

2 Preliminary

Definition 2.1. Let $T$ and $S$ be self maps of a set $X$. Maps $T$ and $S$ are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 2.2. Let $T$ and $S$ be self maps of a set $X$. If $w = Tx = Sx$ for some $x \in X$, then $x$ is called a coincidence point of $T$ and $S$, and $w$ is called a point of coincidence of $T$ and $S$.

Example 2.3. Take $X = [0, 1]$, $Sx = x^2$, $Tx = \frac{x}{2}$. It is clear that $\{0, \frac{1}{2}\}$ is the set of coincidence points of $S$ and $T$ and $0$ is the unique common fixed point.

Definition 2.4. The mappings $S$ and $T$ are said to be weakly compatible if and only if they commute at their coincidence points.

Definition 2.5. The mappings $S$ and $T$ are said to be occasionally weakly compatible if and only if they commute at some coincidence point of $S$ and $T$, i.e. $STu = TSu$ for some coincidence point $u$.

Lemma 2.6. If a weakly compatible pair $(S,T)$ of self maps has a unique point of coincidence, then the point of coincidence is a unique common fixed point of $S$ and $T$.

Example 2.7. Take $X = [0, 1]$, $Sx = x^2$, $Tx = \frac{x}{2}$. It is clear that $\{0, \frac{1}{2}\}$ is the set of coincidence points of $S$ and $T$, $ST0 = TS0$ but $ST\frac{1}{2} \neq TS\frac{1}{2}$ and so $S$ and $T$ are occasionally weakly compatible but not weakly compatible.

Definition 2.8. Let $X$ be a non-empty set, and suppose the mapping $d : X \times X \rightarrow X$ is said to be a metric space if it satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$. 
for all distinct \( x, y, z \in X \).

**Example 2.9.** Let \( E = \mathbb{R}^2 \), \( P = \{(x, y) \in E : x, y \geq 0\} \), \( X = \mathbb{R} \) and \( d : X \times X \rightarrow X \) defined by

\[
d(x, y) = (|x - y|, \alpha |x - y|)
\]

where \( \alpha \geq 0 \) is a constant. Then \((X, d)\) is a metric space.

**Definition 2.10.** Let \((X, d)\) be a complete metric space, if every Cauchy sequence is convergent in \( X \).

**Definition 2.11.** Implicit function: Let \( \mathfrak{F} \) denote the family of all continuous functions \( \varphi : [0, 1]^4 \rightarrow \mathbb{R} \) satisfying the following conditions:

- \((F1)\) For every \( u > 0 \), \( v \geq 0 \) with \( \varphi(u, v, u, v) \geq 0 \) or \( \varphi(u, v, v, u) \geq 0 \), we have \( u < v \).

- \((F2)\) \( \varphi(u, u, 0, 0) < 0 \), for all \( u > 0 \).

**Example 2.12.** Define \( \varphi : [0, 1]^4 \rightarrow \mathbb{R} \) as \( \varphi(t_1, t_2, t_3, t_4) = t_1 - \delta(\max\{t_2, t_3, t_4\}) \), where \( \delta : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( \delta(s) > s \) for \( 0 < s < 1 \). Then

- \((F1)\) \( \varphi(u, v, u, v) = u - \delta(\max\{v, v, v\}) \geq 0 \). If \( u \geq v \), then \( u - \delta(u) \geq 0 \) imply \( u \geq \delta(u) > u \), a contradiction. Hence \( u < v \).

- \((F2)\) \( \varphi(u, u, 0, 0) = u - \delta(\max\{u, 0, 0\}) = u - \delta(u) < 0 \), \( \forall u > 0 \).

**Example 2.13.** Define \( \varphi : [0, 1]^4 \rightarrow \mathbb{R} \) as \( \varphi(t_1, t_2, t_3, t_4) = t_1 - \alpha \max\{t_2, t_3, t_4\} \), where \( \alpha > 1 \). Then

- \((F1)\) \( \varphi(u, v, u, v) = u - \alpha \max\{v, v, v\} \geq 0 \). If \( u \geq v \), then \( u - \alpha u \geq u \), a contradiction. Hence \( u < v \).

- \((F2)\) \( \varphi(u, u, 0, 0) = u - \alpha \max\{u, 0, 0\}) = u(1 - \alpha) < 0 \), \( \forall u > 0 \).

### 3 Main results

**Theorem 3.1.** Let \( T_1, T_2, T_3 \) and \( T_4 \) be four self-mappings of a metric space \((X, d)\) satisfying the condition:

\[
\varphi(d(T_1 x, T_2 y), d(T_3 x, T_4 y), d(T_3 x, T_1 x), d(T_2 y, T_4 y)) \geq 0
\]

for all distinct \( x, y \in X \) where \( \varphi \in \mathfrak{F} \). If \( T_1(X) \subset T_1(X), T_2(X) \subset T_2(X) \) and one of \( T_1(X), T_2(X), T_3(X) \) or \( T_4(X) \) is a complete subspace of \( X \). Then pair \((T_1, T_3)\) and \((T_2, T_4)\) has a point of coincidence. Moreover, if the pairs \((T_1, T_2)\) and \((T_3, T_4)\) are weakly compatible, then \( T_1, T_2, T_3 \) and \( T_4 \) have a unique common fixed point.
Proof. Let \( x_0 \) be an arbitrary point in \( X \). Construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = T_3 x_{2n+1} = T_1 x_{2n} \quad \text{and} \quad y_{2n+1} = T_3 x_{2n+2} = T_2 x_{2n+1}.
\]

The sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) are such that \( x_n \to x, y_n \to y \), implies \( d(x_n, y_n) \to d(x, y) \). Now,

\[
\varphi(d(T_1 x_{2n}, T_2 x_{2n+1}), d(T_3 x_{2n}, T_4 x_{2n+1}), d(T_3 x_{2n}, T_1 x_{2n}), d(T_2 x_{2n+1}, T_4 x_{2n+1})) \geq 0
\]

or

\[
\varphi(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})) \geq 0.
\]

Hence in view of (F1), we have \( d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n}) \). Thus \( \{d(y_{2n}, y_{2n+1})\} \), \( n > 0 \) is a decreasing sequence of positive real numbers in \([0, 1]\) and therefore tends to \( 0 \). Therefore for every \( n \in N \), using analogous arguments one can also show that \( \{d(y_{2n+1}, y_{2n+2})\}, n \geq 0 \) is a sequence of positive real numbers in \([0, 1]\) which converges to \( 0 \). Therefore for every \( n \in N \)

\[
d(y_{n}, y_{n+1}) < d(y_{n-1}, y_{n}) \quad \text{and} \quad d(y_{n}, y_{n+1}) \to 0.
\]

Now for any positive integer \( p \),

\[
d(y_{n}, y_{n+p}) \leq d(y_{n}, y_{n+p-1}) + \ldots + d(y_{n+p-1}, y_{n+p}).
\]

it follows that \( d(y_{n}, y_{n+p}) \to 0 \) which shows that \( \{y_n\} \) is a Cauchy sequence in \( X \). Now suppose that \( S(X) \) is a complete subspace of \( X \), then the subsequence \( \{y_{2n+1}\} \) must converge in \( S(X) \). Call this limit to be \( u \). Then \( Su = u \). As \( \{y_n\} \) is a Cauchy sequence containing a convergent subsequence \( \{y_{2n+1}\} \), therefore the sequence \( \{y_n\} \) also converges implying thereby the convergence of \( \{y_{2n}\} \) being a subsequence of the convergent sequence \( \{y_n\} \). If \( T_1 v \neq T_3 v \), then on setting \( x = v \) and \( y = x_{2n+1} \) we get

\[
\varphi(d(T_1 v, T_2 x_{2n+1}), d(T_3 v, T_4 x_{2n+1}), d(T_3 v, T_1 v), d(T_2 x_{2n+1}, T_4 x_{2n+1})) \geq 0
\]

which on letting \( n \to \infty \) reduces to

\[
\varphi(d(T_1 v, u), d(T_3 v, u), d(T_3 v, T_1 v), d(u, u)) \geq 0
\]

\[
\varphi(d(T_1 v, T_3 v), 0, d(T_3 v, T_1 v), 0) \geq 0
\]

There for \( d(T_1 v, T_3 v) < 0 \), a contradiction. Hence, \( T_1 v = T_3 v \) which shows that the pair \( (T_1, T_3) \) has a point of coincidence.

As \( T_1(X) \subset T_4(X) \) and \( T_1 v = u \) implies that \( u \in T_4(X) \).

Let \( w \in T_4^{-1} u \), then \( T_4 w = u \). Suppose that \( T_4 w \neq T_2 w \).

By using (1), we have

\[
\varphi(d(T_1 x_{2n}, T_2 w), d(T_3 x_{2n}, T_4 w), d(T_3 x_{2n}, T_1 x_{2n}), d(T_2 w, T_4 w)) \geq 0
\]
which on letting \( n \to \infty \) reduces to \( \varphi(d(T_4w, T_2w), 0, 0, d(T_4w, T_2w)) \geq 0 \) which implying, \( d(T_4w, T_2w) < 0 \), a contradiction. Hence \( T_4w = T_2w \). Thus we have \( u = T_1v = T_3v = T_2w = T_4w \) which amounts to say that both the pairs have point of coincidence. If we assumes \( T(X) \) to be complete, then analogous arguments establish this claim. The remaining two cases pertain essentially to the previous cases. Indeed, if \( T_1(X) \) is complete then \( u \in T_1(X) \subset T_4(X) \) and if \( T_2(X) \) is complete then \( u \in T_2(X) \subset T_3(X) \). Moreover, if the pairs \( (T_1, T_3) \) and \( (T_2, T_4) \) are weakly compatible at \( v \) and \( w \) respectively, then \( T_1u = T_1(T_3v) = T_3(T_1v) = T_3u \) and \( T_2u = T_2(T_4w) = T_4(T_2w) = T_4u \). If \( T_1u \neq u \), then

\[
\varphi(d(T_1u, T_2w), d(T_3u, T_4w), d(T_3u, T_1u), d(T_2w, T_4w)) \geq 0
\]

which contradicts (F2). Hence \( T_1u = u \). Similarly we can show that \( T_2u = u \). Thus \( u \) is a common fixed point of \( T_1, T_2, T_3 \) and \( T_4 \). The uniqueness of common fixed point follows easily. Also \( u \) remains the unique common fixed point of both the pairs separately. This completes the proof.

**Corollary 3.2.** Let \( S \) and \( T \) be two self-mappings of a metric space \( (X, d) \) satisfying the condition:

\[
\varphi(d(Sx, Sy), d(Tx, Ty), d(Tx, Sx), d(Sy, Ty)) \geq 0
\]

for all \( x, y \in X \), where \( \varphi \in \mathfrak{F} \). If \( S(X) \subset T(X) \) and one of \( S(X) \) and \( T(X) \) is complete subspace of \( X \). Then the pair \( (S, T) \) has a point of coincidence. Moreover, if the pair \( (S, T) \) is weakly compatible, then \( S \) and \( T \) have a unique common fixed point.

**Proof.** The proof of this corollary follows by setting \( T_2 = T_1 = S \) and \( T_4 = T_3 = T \) in the above theorem. \( \square \)

**A Class of Implicit Relation:**

**Definition 3.3.** Let \( \mathfrak{F} \) denote the family of all continuous functions \( \psi : [0, 1]^4 \to R \) satisfying the following conditions:

\( (G1) \) For every \( u > 0, v \geq 0 \) with \( \psi(u, v, v, u) \geq 0 \) or \( \psi(u, v, v, v) \geq 0 \) or \( \psi(u, v, v, 0) \geq 0 \) we have \( u \leq v \).

\( (G2) \) \( \psi \) is decreasing in 2nd, 3rd and 4th variable.

**Example 3.4.** Define \( \psi : [0, 1]^4 \to R \) as \( \psi(t_1, t_2, t_3, t_4) = t_1 - \max\{t_2, t_3, t_4\} \), where \( \delta : [0, 1] \to [0, 1] \) is a continuous function such that \( \delta(s) > s \) for \( 0 < s < 1 \).

**Theorem 3.5.** Let \( (X, d) \) be a complete metric space and let \( T_1, T_2, T_3 \) and \( T_4 \) be the self mappings of \( X \), satisfying the following condition
In general, we get sequence

\[ \psi \left( \frac{d(T_1 x, T_2 y)}{k}, \frac{d(T_3 x, T_1 x)}{2}, d(T_4 y, T_3 x), \frac{d(T_4 y, T_2 y)}{k} \right) \geq 0 \quad \ldots (1) \]

For some \( \psi \in \mathfrak{F} \), every \( x, y \in X \), then \( T_1, T_2, T_3 \) and \( T_4 \) have a unique common fixed point in \( X \).

Proof. Let \( x_0 \) be any arbitrary point. Since \( T_1(X) \subseteq T_4(X) \), \( T_2(X) \subseteq T_3(X) \).
So there must exists points \( x_1, x_2 \in X \) such that \( T_1 x_0 = T_4 x_1 \) and \( T_2 x_1 = T_3 x_2 \).
In general, we get sequence \( \{y_n\} \) and \( \{x_n\} \) in \( X \) such that \( y_{2n} = T_4 x_{2n+1} = T_1 x_{2n} \) and \( y_{2n+1} = T_3 x_{2n+2} = T_2 x_{2n+1} \).

\[ 0 \leq \psi \left( \frac{d(T_1 x_{2n}, T_2 x_{2n+1})}{k}, \frac{d(T_3 x_{2n}, T_1 x_{2n}) + d(T_4 x_{2n+1}, T_3 x_{2n})}{2}, d(T_4 x_{2n+1}, T_3 x_{2n}), \frac{d(T_4 x_{2n+1}, T_2 x_{2n+1})}{k} \right) \]

\[ \psi \left( \frac{d(y_{2n}, y_{2n+1})}{k}, \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{2}, d(y_{2n}, y_{2n-1}), \frac{d(y_{2n}, y_{2n+1}+1)}{k} \right) \geq 0 \]

\[ \psi \left( \frac{d(y_{2n}, y_{2n+1})}{k}, \frac{d(y_{2n-1}, y_{2n})}{2}, d(y_{2n}, y_{2n-1}), \frac{d(y_{2n}, y_{2n+1}+1)}{k} \right) \geq 0 \]

\[ \psi \text{ is non- increasing function,} \]

\[ \psi \left( \frac{d(y_{2n}, y_{2n+1})}{k}, d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), \frac{d(y_{2n}, y_{2n+1}+1)}{k} \right) \geq 0 \]

\[ \Rightarrow d(y_{2n}, y_{2n+1}) \leq \frac{d(y_{2n}, y_{2n+1})}{k}. \text{ Similarly } d(y_{2n+1}, y_{2n+2}) \leq \frac{d(y_{2n+1}, y_{2n+2})}{k}. \]

There fore for all \( n \) we get \( d(y_n, y_{n+1}) \leq \frac{d(y_1, y_2)}{k^n} \).

Hence \( d(y_n, y_{n+1}) \leq \frac{d(y_{n-1}, y_n)}{k} \leq \frac{d(y_{n-2}, y_{n-1})}{k^2} \leq \ldots \leq \frac{d(y_0, y_1)}{k^n} \).

There fore \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \)

Now for any positive integer \( p \),

\[ d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + \ldots + d(y_{n+p-1}, y_{n+p}). \]

It follows that \( d(y_n, y_{n+p}) \to 0 \) which shows that \( \{y_n\} \) is a Cauchy sequence in \( X \) which is complete. Therefore \( \{y_n\} \) converges to \( z \), for some \( z \in X \). So it follows that \( \{T_1 x_{2n}\}, \{T_3 x_{2n}\}, \{T_2 x_{2n+1}\} \) and \( \{T_4 x_{2n+1}\} \) also converges to \( z. \)
That is, there exists a finite sequence
Now if \( T_4(X) \) is complete then, if we take \( z \in T_4(X) \), so there exist \( u \in X \), such that \( z = T_4 u \)

\[ \psi \left( \frac{d(T_1 x_{2n}, T_2 u)}{k}, \frac{d(T_3 x_{2n}, T_1 x_{2n}) + d(T_1 x_{2n}, T_4 u)}{2}, d(T_4 u, T_3 x_{2n}), \frac{d(T_4 u, T_2 u)}{k} \right) \geq 0. \]
As \( n \to \infty \)
\[
\psi \left( \frac{d(z, T_2 u)}{k}, \frac{d(z, z) + d(z, z)}{2}, d(z, z), \frac{d(z, T_2 u)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(z, T_2 u)}{k}, 0, 0, \frac{d(z, T_2 u)}{k} \right) \geq 0. \text{ So } d(z, T_2 u) \leq 0.
\]
Hence \( z = T_2 u = T_3 u = T_4 u \). Now \((T_2, T_3)\) is weakly compatible, so \( T_2 T_3 u = T_3 T_2 u \), there by \( T_2 z = T_3 z \).

Now, \( x = x_{2n} \) and \( y = z \) in (1) we get,
\[
\psi \left( \frac{d(T_1 x_{2n}, T_2 z)}{k}, \frac{d(T_3 x_{2n}, T_1 x_{2n}) + d(T_1 x_{2n}, T_4 z)}{2}, d(T_4 z, T_3 x_{2n}), \frac{d(T_4 z, T_2 z)}{k} \right) \geq 0.
\]
Taking limit as \( n \to \infty \)
\[
\psi \left( \frac{d(z, T_2 z)}{k}, \frac{d(z, z) + d(z, T_4 z)}{2}, d(T_4 z, z), \frac{d(T_4 z, T_2 z)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(z, T_2 z)}{k}, \frac{d(z, T_4 z)}{2}, d(T_4 z, z), \frac{d(T_4 z, T_2 z)}{k} \right) \geq 0.
\]
Since \( z = T_4 z, z \in T_4(X) \), so we have
\[
\psi \left( \frac{d(z, T_2 z)}{k}, \frac{d(z, z)}{2}, d(z, z), \frac{d(z, T_2 z)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(z, T_2 z)}{k}, 0, 0, \frac{d(z, T_2 z)}{k} \right) \geq 0.
\]
Since \( \psi \) is non- increasing. Hence \( T_2 z = T_4 z = z \).

As \( T_2(X) \subseteq T_4(X) \) Let there exists \( v \in X \), such that \( z = T_2 z = T_3 v \). Now put \( x = v, y = z \) in (1)
\[
\psi \left( \frac{d(T_1 v, T_2 z)}{k}, \frac{d(T_3 v, T_1 v) + d(T_1 v, T_4 z)}{2}, d(T_4 z, T_3 v), \frac{d(T_4 z, T_2 z)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(T_1 v, z)}{k}, \frac{d(T_3 v, T_1 v) + d(T_1 v, z)}{2}, d(z, T_3 v), \frac{d(z, z)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(T_1 v, z)}{k}, \frac{d(z, T_1 v) + d(T_1 v, z)}{2}, d(z, z), \frac{d(z, z)}{k} \right) \geq 0.
\]
\[
\psi \left( \frac{d(T_1 v, z)}{k}, d(z, T_1 v), 0, 0 \right) \geq 0.
\]
This implies that \( T_1 v = z \). Now since \( T_1 \subseteq T_4 \) so \( z = T_1 v \in T_4 \). Therefore \( z = T_1 v = T_4 v \). Now as \((T_1, T_4)\) is weakly compatible so \( T_1 T_4 v = T_4 T_1 v \) such that \( T_1 z = T_4 z \). So combining all the results, we have \( T_1 z = T_4 z = T_2 z = T_3 z = z \).
Similarly we can show that \( T_1 z = T_4 z = T_2 z = T_3 z = z \) by taking \( T_3(X) \) is complete. Thus \( z \) is the common fixed point of \( T_1, T_2, T_3 \) and \( T_4 \).

**Uniqueness:** Let \( p \) and \( z \) be the two common fixed points of maps \( T_1, T_2, T_3 \) and \( T_4 \). Put \( x = z \) and \( y = p \) in condition (1) we get

\[
\psi \left( \frac{d(T_1 z, T_2 p)}{k}, \frac{d(T_3 z, T_1 z) + d(T_1 z, T_4 p)}{2}, d(T_4 p, T_3 z), \frac{d(T_4 p, T_2 p)}{k} \right) \geq 0
\]

\[
\psi \left( \frac{d(z, p)}{k}, \frac{d(z, z) + d(z, p)}{2}, d(p, z), \frac{d(p, p)}{k} \right) \geq 0
\]

\[
\psi \left( \frac{d(z, p)}{k}, \frac{d(z, z)}{2}, d(p, z), 0 \right) \geq 0
\]

Hence \( p = z \). So \( z \) is the unique common fixed points of \( T_1, T_2, T_3 \) and \( T_4 \).

**Corollary 3.6.** Let \((X, d)\) be a complete metric space and let \( S \) and \( T \) be the self mapping of \( X \), satisfying the following condition

\[(2) \text{ The pairs } (S, T) \text{ are weakly compatible.}\]

\[(3) \text{ } S(X) \text{ or } T(X) \text{ is complete.}\]

\[(4) \text{ There exists } k \in (0, 1) \text{ such that}\]

\[
\psi \left( \frac{d(S x, S y)}{k}, \frac{d(T x, S x) + d(S x, T y)}{2}, d(T y, T x), \frac{d(T y, S y)}{k} \right) \geq 0 \quad ......(1)
\]

For some \( \psi \in \mathcal{F} \), every \( x, y \in X \), then \( S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** The proof follows by taking \( T_3 = T_4 = T \) and \( T_1 = T_2 = S \) in the above theorem.

**References**


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