The Variance of Fractional Partial Differential Equation of Distributed Order

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Abstract

The aim of this paper is to express variance of fractional partial differential equation of distributed order, where the Caputo fractional derivative is considered of order $0 < \alpha \leq 1$. This expression is achieved by using a function transform, Fourier and Laplace transforms. In particular the asymptotic behavior of the fundamental solution and its variance are studied. As special case the variance of the fundamental solution of time fractional diffusion equation is obtained.

Keywords: Fractional derivatives, Laplace transform, Fourier transform, variance, distributed order, Caputo fractional derivative

1. Introduction

The partial equation of Gaussian diffusion is generalized by using the time fractional derivative of distributed order, where the order of this type is considered in the Caputo sense[4,7,8,9].

In this paper we will take fractional advection dispersion equation and fractional diffusion equation as two forms of fractional partial differential equation. Also we derive some results concerning variance and asymptotic behaviors of fractional advection dispersion equation of distributed order.
Then we will study how Mainardi and Pagnini [13] and Mainardi, Pagnini and Gorenflo [12] derived the variance and asymptotic behaviors of the solution of fractional diffusion equation of distributed order as special case of fractional advection dispersion equation.

The plan of the paper is as follows:

In section 2, we write down the general form of the time fractional advection dispersion equation with distributed order with Caputo derivative and Fourier-Laplace representation of the corresponding fundamental solution. For this purpose we need to introduce a positive function \( w(\beta) \) that acts as a discrete or continuous distribution of orders. We study the fractional advection dispersion equation of distributed order as a discrete distribution, so we take two distinct orders \( \beta_1, \beta_2 : 0 < \beta_1 < \beta_2 \leq 1 \).

Section 2.3 is devoted to the time evolution of variance which is obtained from the Fourier-Laplace representation of the corresponding fundamental solution, by inverting only the Laplace transform. We well visible from their asymptotic expressions for small and large times.

Section 3, we take as special case the variance of the fundamental solution of the time fractional diffusion equation of distributed order.

\section{The Fractional Advection Dispersion Equation of Distributed Order:}

The fractional advection-dispersion equation of single order is written as

\[
\frac{\partial^\beta C(x,t)}{\partial t^\beta} = D \frac{\partial^2 C(x,t)}{\partial x^2} - b \frac{\partial C(x,t)}{\partial x} + \lambda C(x,t) \tag{2.1}
\]

where \( x > 0, \ t > 0, \ 0 < \beta \leq 1 \) and \( \frac{\partial^\beta}{\partial t^\beta} \) is the fractional derivative of Caputo type; \( D > 0, \ b \geq 0 \) and \( \lambda \geq 0 \). The initial conditions is assumed to be

\[
C(x,0^+) = C_0(x) \tag{2.2}
\]

Eq.(2.1) can be generalized by using the notion of fractional derivative of distributed order in time. For this purpose, we need to consider a function \( w(\beta) \) that acts as a weight function for the order of differentiation \( \beta \in (0,1] \) such that

\[
w(\beta) \geq 0 \quad \text{and} \quad \int_0^1 w(\beta) d\beta = 1 \tag{2.3}
\]

The time fractional advection-dispersion equation of distributed order can be written as

\[
\int_0^1 w(\beta) \frac{\partial^\beta C(x,t)}{\partial t^\beta} d\beta = D \frac{\partial^2 C(x,t)}{\partial x^2} - b \frac{\partial C(x,t)}{\partial x} + \lambda C(x,t) \tag{2.4}
\]

Subjected to initial conditions (2.2).
Note that in case \( b = \lambda = 0 \) and \( D = 1 \), then Eq.(2.4) reduces to the diffusion equation of fractional distributed order (2.4) with \( C(x,t) = u(x,t) \).

Now we assume the function transform
\[
C(x,t) = u(\zeta,t)\exp(\mu\zeta), \quad \zeta = \frac{x}{\sqrt{D}}, \quad \mu = \frac{b}{2\sqrt{D}},
\]
then Eq.(2.4) reduces to
\[
\int_0^1 w(\beta) \frac{\partial^\beta u(\zeta,t)}{\partial t^\beta} d\beta = \frac{\partial^2 u(\zeta,t)}{\partial \zeta^2} - \theta^2 u(\zeta,t) \tag{2.5}
\]
where \( \theta^2 = \mu^2 + \lambda \). Assuming that \( C(x,0^+) = \delta(x)\exp(b/2D) \), where \( \delta(x) \) is the Dirac-delta function, then \( u(\zeta,0^+) = \delta(\zeta) \) is the initial condition of Eq.(2.5).

2.1 The Caputo Form in Fourier-Laplace domain:

The fundamental solution of the time-fractional advection dispersion equation (2.5) can be obtained by applying in sequence the Fourier and Laplace transforms to them. The researcher write, for generic function \( f(x) \), these transforms [2]:
\[
\mathcal{F}\{f(x); \kappa\} = \hat{f}(\kappa) := \int_{-\infty}^{\infty} e^{ixx} f(x) dx \quad ; \kappa \in \mathbb{R}
\]
\[
\mathcal{L}\{f(x); p\} = \tilde{f}(p) := \int_{-\infty}^{\infty} e^{-px} f(t) dt \quad ; p \in D
\]
Now applying the Laplace transform with respect to \( t \), then Eq.(2.5) implies
\[
\int_0^1 \left[ w(\beta) p^\beta \hat{u}(\zeta,p) - p^{\beta-1} u(\zeta,0) \right] d\beta = \frac{\partial^2 \hat{u}(\zeta,p)}{\partial \zeta^2} - \theta^2 \hat{u}(\zeta,p) \tag{2.6}
\]
And applying the Fourier transform with respect to \( \zeta \) yields
\[
\left[ \int_0^1 w(\beta) p^\beta d\beta \right] \hat{u}(k,p) = \left[ \int_0^1 \hat{u}(k,0) (k^2 + \theta^2) \right] \hat{u}(k,p)
\]
which reduces to
\[
\hat{u}(k,p) = \frac{B(p)/p}{B(p) + k^2 + \theta^2} \tag{2.7}
\]
where \( B(p) = \int_0^1 w(\beta) p^\beta d\beta \) and \( \hat{u}(k,0) = \hat{\delta}(k) = 1 \).

2.2 The variance of the fundamental solution:

In this section, we worth to outline the expression of the second moment (variance) of the fundamental solution of the fractional advection dispersion equation of distributed order by a single Laplace inversion, as it is shown hereafter.

Denoting for the form
we easily recognize that
\[ \sigma^2(t) = -\frac{\partial^2}{\partial \kappa^2} \tilde{u}(\kappa = 0, t) \]
As a consequence the researcher need to invert only the Laplace transforms taking into account the behavior of the Fourier transform for \( \kappa \) near zero, we get from equation (2.8)
\[ \tilde{\sigma}^2(p) = \frac{-\partial^2 \tilde{u}(\kappa = 0, p)}{\partial \kappa^2} = \frac{2B(p)}{p(B(p) + \theta)^2} \]  
(2.9)

### 2.3 Fractional advection dispersion equation of double order:
The expected sub-linear growth with time is shown in the following special case of \( B(\beta) \) treated in [3].
Now, let us assume the double order case of the second moment by assuming that
\[ w(\beta) = w_1 \delta(\beta - \beta_1) + w_2 \delta(\beta - \beta_2), \]
where \( 0 < \beta_1 < \beta_2 \leq 1, \ w_1 > 0, \ w_2 > 0, \ w_1 + w_2 = 1 \). Then \( B(p) \) is written as
\[ B(p) = w_1 p^{\beta_1} + w_2 p^{\beta_2} \]  
(2.11)
Substituting into Eq.(2.9), we get
\[ \tilde{\sigma}^2(p) = \frac{2(w_1 p^{\beta_1 - 1} + w_2 p^{\beta_2 - 1})}{(w_1 p^{\beta_1} + w_2 p^{\beta_2} + \theta^2)^2} \]
\[ \sim \left\{ \begin{array}{ll}
2w_1 p^{\beta_1 - 1} & , p \to 0^+ \\
2w_2 p^{\beta_2 - 1} & , p \to \infty \\
(w_1 p^{\beta_1} + \theta^2)^2 & 
\end{array} \right. \]  
(2.13)
Inverting via the inverse Laplace transform by making use of the formula
\[ t^{-1}\{ \frac{n! p^{\lambda-\eta}}{(p+c)^{n+1}}, t \} = t^{\lambda+\eta-1} E_{\lambda,\eta}^{(n)}(-ct^\lambda) \]  
(2.14)
And applying the Tauberian theory, we obtain
\[ \sigma^2(t) \sim \left\{ \begin{array}{ll}
\frac{2t^{\beta_1} E_{\beta_1}^{(1)}(-\theta^2 t^{\beta_1})}{w_1} & , t \to 0^+ \\
\frac{2t^{\beta_2} E_{\beta_2}^{(1)}(-\theta^2 t^{\beta_2})}{w_2} & , t \to \infty \\
\end{array} \right. \]  
(2.15)
where \( E_{\beta}(t) \) is the well-known Mittag-Leffler function [1,5,6,14], and \( E_{\beta}^{(1)}(t) \) is its first derivative with respect to \( t \), which formula is
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\[ E^{(1)}_{\beta_i} \left( -\frac{\theta^2}{w_i} t^\beta_i \right) = \sum_{j=0}^{\infty} \frac{(j+1)(-\theta^2)^j}{j! \Gamma(\beta_i j + \beta_i + 1)} \]  

(2.16)

which for \( \theta = 0 \) terminates for all terms except when \( j = 0 \), and in this case it is reduced to \( 1/\Gamma(\beta_i + 1) \). Also \( E^{(1)}_{\beta_i} \left( -\frac{\theta^2}{w_i} t^\beta_i \right) \) is reduced to \( 1/\Gamma(\beta_i + 1) \) when \( \theta = 0 \).

3. The time fractional diffusion equation of distributed order:

The fractional diffusion Equation

\[ \frac{\partial^\beta}{\partial t^\beta} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R}, \quad t \geq 0, \quad 0 < \beta \leq 1 \]  

(3.1)

is a special case of Fractional advection dispersion equation by taking \( \theta = 0 \) in equation (2.5). We now consider the so-called time-fractional diffusion equation of distributed order, see [10-12].

\[ \int_0^1 w(\beta) \left[ \frac{\partial^\beta}{\partial t^\beta} u(x,t) \right] d\beta = \int_0^1 \frac{\partial^2 u(x,t)}{\partial x^2} d\beta, \quad b(\beta) \geq 0 \]  

(3.2)

since

\[ \int_0^1 w(\beta) d\beta = 1 \]  

(3.3)

with \( x \in \mathbb{R}, \quad t \geq 0, \quad w(\beta) \) is a weight function. Then

\[ \int_0^1 w(\beta) \left[ \frac{\partial^\beta}{\partial t^\beta} u(x,t) \right] d\beta = \frac{\partial^2 u(x,t)}{\partial x^2} \]  

(3.4)

Now we will take the Laplace transform of Eq.(3.4) as

\[ \int_0^1 \left[ p\beta \hat{u}(x,p) - p^{\beta-1} u(x,0) \right] d\beta = \frac{\partial^2 \hat{u}(x,p)}{\partial x^2} \]  

(3.5)

but since \( u(x,0) = \delta(x) \), then Eq.(1.5) leads to

\[ B(p) \hat{u}(x,p) - \frac{B(p)}{p} \delta(x) = \frac{\partial^2 \hat{u}(x,p)}{\partial x^2} \]  

(3.6)

where

\[ B(p) = \int_0^1 w(\beta) p^\beta d\beta \]  

(3.7)

The Fourier integral transform of Eq.(3.6) with respect to \( x \) can be written as

\[ B(p) \hat{u}(\kappa,p) - \frac{B(p)}{p} \hat{\delta}(\kappa) = -\kappa^2 \hat{u}(\kappa,p) \]  

(3.8)
but since $\hat{\delta}(\kappa) = 1$. Then Eq.(3.8) yields

$$\hat{u}(\kappa, p) = \frac{B(p)}{B(p)+\kappa^2}, \quad \text{Re}(p) > 0, \quad \kappa \in \mathbb{R}$$

(3.9)

Before trying to get the solution in the space-time domain, it is worth to outline the expression of its second moment as it can be derived from Eq. (3.9), so using the formula

$$\sigma^2(t) = 2 \frac{t^0}{\Gamma(\beta + 1)} = -\frac{\partial^2 \hat{u}(\kappa = 0, t)}{\partial \kappa^2}, \quad 0 < \beta \leq 1$$

(3.10)

Then

$$\frac{\partial \hat{u}(\kappa, p)}{\partial \kappa} = -2\kappa \left( \frac{B(p)}{p} \right) \left( \frac{1}{B(p)+\kappa^2} \right)^2$$

(3.11)

and

$$\frac{\partial^2 \hat{u}(\kappa, p)}{\partial \kappa^2} = -\frac{2B(p)/p}{(B(p)+\kappa^2)^4} + 4\kappa^2(B(p)+\kappa^2)B(p)/p$$

(3.12)

Then, from (3.14) the researcher is allowed to drive the asymptotic behaviors of $\sigma^2(t)$ for $t \to 0^+$ and $t \to +\infty$ from the asymptotic behaviors of $B(p)$ for $p \to \infty$ and $p \to 0$, respectively.

The expected sub-linear growth with time is shown in the following special case of $b(\beta)$ treated in [3,12].

This case is slow diffusion (power-law growth) where

$$w(\beta) = w_1\delta(\beta - \beta_1) + w_2\delta(\beta - \beta_2),$$

(3.15)

where $0 < \beta_1 < \beta_2 \leq 1, \quad w_1 > 0, \quad w_2 > 0, \quad w_1 + w_2 = 1$

In fact

$$B(p) = \int_{0}^{1} w(\beta) p^\beta d\beta = \int_{0}^{1} \left[ w_1\delta(\beta - \beta_1) + w_2\delta(\beta - \beta_2) \right] p^\beta$$

$$= \int_{0}^{1} w_1 p^{\beta_1} d\beta + \int_{0}^{1} w_2 p^{\beta_2} d\beta = w_1 p^{\beta_1} + w_2 p^{\beta_2}$$

(3.16)
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\[ \delta (\beta - \beta_i) = \begin{cases} 0 & \beta \neq \beta_i, \\ 1 & \beta = \beta_i \end{cases}, \quad \delta (\beta - \beta_2) = \begin{cases} 0 & \beta \neq \beta_2, \\ 1 & \beta = \beta_2 \end{cases} \]  

Substitute (3.17) in Eq.(3.14) we get

\[ \tilde{\sigma}^2(p) = \frac{2}{w_1 p^{\beta_i + 1} + w_2 p^{\beta_2 + 1}} \sim \begin{cases} \frac{2}{w_2 p^{\beta_2 + 1}}, & p \to \infty \\ \frac{2}{w_1 p^{\beta_i + 1}}, & p \to 0^+ \end{cases} \]  

So the inversion Laplace transform of asymptotic formula (3.18) is

\[ \sigma^2(t) \sim \begin{cases} \frac{2}{w_2 \Gamma(\beta_2 + 1)} t^{\beta_2}, & t \to 0^+ \\ \frac{2}{w_1 \Gamma(\beta_1 + 1)} t^{\beta_1}, & t \to \infty \end{cases} \]  

This results is taken from [3,12], and it’s a special case of results of fractional advection dispersion equation which we achieved.

4. Conclusion:

It is concluded to express variance of fractional partial differential equation of distributed order and asymptotic behaviors of \( \sigma^2(t) \).

References


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