An Extension of Fuglede-Putnam Theorem for Certain Posinormal Operators

A. Bachir

Department of Mathematics, Faculty of Science
King Khalid University, Abha, P.O.Box 9004, Saudi Arabia
bachir_ahmed@hotmail.com

Abstract. An asymmetric Fuglede-Putnam Theorem for certain posinormal operators is proved. As a consequence of this result, we obtain that the range of the generalized derivation induced by these classes of operators is orthogonal to its kernel.

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1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space, and let $B(\mathcal{H})$, $C_2$ and $C_1$ denote the algebra of all bounded linear operators on $\mathcal{H}$, the Hilbert Schmidt class and the trace class in $B(\mathcal{H})$ respectively. It's well known that $C_2(\mathcal{H})$ and $C_1(\mathcal{H})$ each form a two-sided $^*$–ideal in $B(\mathcal{H})$ and $C_2(\mathcal{H})$ is
its self a Hilbert space with the inner product
\[ \langle X, Y \rangle = \sum \langle X e_i, Y e_i \rangle = tr(Y^*X) = tr(XY^*), \]
where \( \{ e_i \} \) is any orthonormal basis of \( \mathcal{H} \) and \( tr(.) \) is the natural trace on \( C_1(\mathcal{H}) \). The Hilbert-Schmidt norm of \( X \in C_2(\mathcal{H}) \) is given by \( \| X \|_2 = \langle X, X \rangle^{\frac{1}{2}} \).

For any operator \( A \in B(\mathcal{H}) \), set, as usual, \( |A| = (A^*A)^{\frac{1}{2}} \) and \( [A^*, A] = A^*A - AA^* = |A|^2 - |A|^2 \) (the self commutator of \( A \)), and consider the following definitions: \( A \) is normal if \( A^*A = AA^* \), hyponormal if \( A^*A - AA^* \geq 0 \), \( p \)-hyponormal if \( |A|^{2p} \geq |A^*|^{2p} \) \((0 < p \leq 1)\). An \( 1 \)-hyponormal is hyponormal which has been studied by many authors and its known that hyponormal operators have many interesting properties similar to those of normal operators [9]. An operator \( A \in B(\mathcal{H}) \) is positive, \( A \geq 0 \), if \( (Ax, x) \geq 0 \) for all \( x \in \mathcal{H} \).

According to [10] an operator \( A \in B(\mathcal{H}) \) is called posinormal if there exists a positive \( P \in B(\mathcal{H}) \) such that \( AA^* = A^*PA \). Here \( P \) is called interrupter of \( A \). In other words, an operator \( A \) is posinormal if \( AA^* \leq c^2 A^*A \), where \( c > 0 \).

An operator \( A \) is said to be \( p \)-posinormal \((0 < p \leq 1)\) if \( (AA^*)^p \leq c^2 (A^*A)^p \) for some \( c > 0 \).

According to [7] an operator is called \((p, k)\)-quasiposinormal if
\[ A^{*k}(c^2(A^*A)^p - (AA^*)^p)A^k \geq 0 \]
for some positive integer \( 0 < p \leq 1 \), some \( c > 0 \) and a positive integer \( k \). These classes are related by proper inclusion

\[ \text{hyponormal} \subset p\text{-hyponormal} \subset (p, k)\text{-quasiposinormal} \]

for a positive integer \( k \) and a positive number \( 0 < p \leq 1 \).

The famous Fuglede-Putnam theorem (see [4, 6]) asserts that if \( A \) and \( B \) are normal operators and \( AX = XB \) for some operator \( X \), then \( A^*X = XB^* \). Several authors have extended this theorem to non normal operators (see [4], [5], [8], [11], [9] and [12]).

Berberian [3] relaxes the hypothesis on \( A \) and \( B \) by assuming \( A \) and \( B^* \) hyponormal operators and \( X \) to be Hilbert-Schmidt class. Patel [11] proved that if \( A \) and \( B^* \) are \( p \)-hyponormal operators such that \( AX = XB \) for \( X \in C_2(\mathcal{H}) \), then \( A^*X = XB^* \).

Let \( A, B \) in \( B(\mathcal{H}) \), we define the generalized derivation \( \delta_{A,B} \) induced by \( A \) and \( B \) as follows
\[ \delta_{A,B}(X) = AX - XB \text{ for all } X \in B(\mathcal{H}). \]
J. Anderson and C. Foias [2] proved that if $A$ and $B$ are normal, $S$ is an operator such that $AS = SB$, then
\[ \|\delta_{A,B}(X) - S\| \geq \|S\|, \text{ for all } X \in B(\mathcal{H}). \]

Where $\|\cdot\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of Definition 1.2 in [1]. The purpose of this paper is to prove that the Fuglede-Putnam theorem remains true if $A$ is hyponormal and $B^*$ is an invertible posinormal operator. As a consequence of this result, we give a similar orthogonality result by proving that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

2. Main results

The basic elementary operator $M_{A,B}$ induced by the operators $A$ and $B$ is defined on $C_2(\mathcal{H})$ by $M_{A,B}(X) = AXB$, and the adjoint of $M_{A,B}$ is given by the formula $M_{A,B}^* = A^*XB^*$.

**Proposition 2.1.** Let $A, B \in B(\mathcal{H})$. If $A \geq 0$ and $B \geq 0$, then $M_{A,B} \geq 0$.

*Proof.* Let $X \in C_2(\mathcal{H})$,
\[
\langle M_{A,B}X, X \rangle = tr(AXBX^*) = tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}) = tr((A^{\frac{1}{2}}X B^{\frac{1}{2}})(A^{\frac{1}{2}}X^* B^{\frac{1}{2}})) \geq 0.
\]

\[ \square \]

**Proposition 2.2.** If $A \in B(\mathcal{H})$ is hyponormal and $B^* \in B(\mathcal{H})$ is posinormal, then $M_{A,B}$ is posinormal.

*Proof.* Let $X \in C_2(\mathcal{H})$, since $B^*$ is posinormal, then there exists $c > 0$ such that $c^2BB^* - B^*B \geq 0$ and
\[
(c^2M_{A,B}^*M_{A,B} - M_{A,B}M_{A,B}^*)(X) = c^2A^*AXB^* - AA^*X^*B
\]
\[
= c^2(A^*AXB^*) - (AA^*X^*B) + c^2AA^*XBB^* - c^2AA^*XBB^*
\]
\[
= c^2(A^*A - AA^*)XBB^* + AA^*X(c^2BB^* - B^*B).
\]

This operator is sum of two positive operators, so positive. Hence $M_{A,B}$ is posinormal. \[ \square \]
Lemma 2.3. If $T \in B(\mathcal{H})$ is an invertible posinormal operator, then $T^{-1}$ is posinormal.

Proof. This proof uses the fact if $A$ is positive operator and $A \geq I$, then $A^{-1} \leq I$. Since $T$ is posinormal, then $c^2T^*T - TT^* \geq 0$ for some $c > 0$. Hence

$$T^{-1}(c^2T^*T)T^{*-1} \geq T^{-1}(TT^*)T^{*-1} = I.$$ 

Taking inverses gives

$$T^{-1}\left(\frac{1}{c^2}T^{-1}T^{*-1}\right) \leq I,$$

and

$$\frac{1}{c^2}T^{-1}T^{*-1} \leq T^{*-1}I^{-1}.$$ 

Thus $T^{-1}T^{*-1} \leq c^2T^{*-1}I^{-1}$. This means that $T^{-1}$ is posinormal. \hfill $\square$

Lemma 2.4. [8] Let $T \in B(\mathcal{H})$ be a $(p,k)-\text{quasiposinormal}$ operator for $0 < p \leq 1$ and a positive integer $k$. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in \mathcal{H}$, then $(T - \lambda)^*x = 0$.

Theorem 2.5. If $A \in B(\mathcal{H})$ is hyponormal operator and $B^* \in B\mathcal{H}$ is an invertible posinormal operator such that $AX = XB$, for some $X$ in $C_2(\mathcal{H})$, then $A^*X = XB^*$.

Proof. Let $AX = XB$ for some $X$ in $C_2(\mathcal{H})$, then $M_{A,B^{-1}}(X) = X$. Since $B^*$ is an invertible posinormal operator, $(B^*)^{-1}$ is a posinormal by Lemma 2.3. Also, $M_{A,B^{-1}}$ is a posinormal operator by Proposition 2.2. Hence $M_{A,B^{-1}}(X) = X$ by Lemma 2.4, that is, $A^*X = XB^*$. This complete the proof. \hfill $\square$

Now we ready to extend the orthogonality result to certain posinormal class operators.

Theorem 2.6. Let $A, B$ be operators in $B(\mathcal{H})$ and $S \in C_2(\mathcal{H})$. Then

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$ 

(2.1)

and

$$\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2$$ 

(2.2)

if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, for all $S \in C_2(\mathcal{H})$. 

Proof. We known that the Hilbert-Schmidt class $C_2(\mathcal{H})$ is a Hilbert space. Note that
\[
\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 + \text{Re}\langle \delta_{A,B}(X), S \rangle
\]
and
\[
\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2 + \text{Re}\langle X, \delta_{A,B}(S) \rangle.
\]
Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ we obtain (2.1) and (2.2). \qed

The claim is verified and the proof is complete.

**Corollary 2.7.** Let $A, B$ be operators in $B(\mathcal{H})$ and $S \in C_2(\mathcal{H})$. Then
\[
\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2
\]
and
\[
\|\delta_{A,B}^*(X) + S\|_2^2 = \|\delta_{A,B}^*(X)\|_2^2 + \|S\|_2^2
\]
if and only if $A$ is hyponormal and $B^*$ is an invertible posinormal operator.

**References**


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