An Approach for Designing a Developable Surface with a Common Geodesic Curve

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Abstract. This paper presents an approach for designing a developable surface possessing a given curve as the geodesic of it. We analyze the necessary and sufficient conditions when the resulting developable surface is a cylinder, cone or tangent surface. Also, we solve the problem of requiring the surfaces of revolution which are developable surfaces. Finally, we illustrate the convenience and efficiency of this approach by some representative examples.

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1 Introduction

Ruled surfaces, particularly developable surfaces, have been widely studied and applied in mathematics and engineering. Intuitively, a developable surface is a surface which can be unfolded (developed) into a plane without stretching
or tearing. Therefore, developable surfaces are attractive in geometric design and representation of product surfaces, owing to the simplicity of the manufacturing process required to fabricate them.

In differential geometry, a ruled surface is generated by sweeping a straight line through 3D space (see, e.g., [1]). This straight line is referred to as a ruling, or generator of the surface. In general, the tangent plane to a ruled surface changes along a ruling. Developable surfaces are subsets of ruled surfaces, where all points of the same generator line share a common tangent plane. The rulings are principal curvature lines with vanishing normal curvature and the Gaussian curvature vanishes at all surface points. It is also known as a single curved surface, as one of its principal curvatures is null. Therefore developable surfaces are also called single-curved surfaces, as opposed to double-curved surfaces.

Many studies on designing with developable surfaces have been reported. Ravani and Ku [2], present some direct representations for developable surfaces and their Bertrand offsets. Aumann [3] proposed the condition under which a developable surface can be constructed with two boundary curves. Lang and Roschel [4] obtained necessary conditions for the control nets and the weights of a rational Bézier surface to become developable. However, the result leads to a complex system of coupled equations that make it difficult to design developable surfaces with their method. Pottmann and Farin [5], Pottmann and Wallner [6] constructed developable surfaces based on the methods of projective geometry. Chu and Séquin [7] and Aumann [8] developed a new approach to geometric design of developable Bézier surfaces based on the de Casteljau algorithm.

The geodesic between two points on a surface is defined as the curve embedded in the surface that connects the points with minimal distance [1]. Geodesic also has been widely applied in various industries, such as tent manufacturing, cutting and painting path, fiberglass tape windings in pipe manufacturing, textile manufacturing ([8-11]). Generally, the research in geodesic focused on how to find and characterize geodesic on the given surfaces ([12]), and there were a lots of papers working on such a problem ([13-16]).

In the designing industry of shoes, garments and so on, it is often required for designers to construct a family of surfaces from a given spatial geodesic curve, among which they can select those satisfying the fashion tastes of customers. This may be regarded as the reverse problem of the above-mentioned. In Ref. [17], Wang et al. studied the problem of constructing a family of surfaces from a given spatial geodesic curve, among which each surface can be a candidate for fashion designing. They derived the necessary and sufficient condition for the coefficients to satisfy both the geodesic and the isoparametric requirements. Based on this representation of the surface, Zhao and Wang [18] proposed a new method for designing a developable surface by constructing a
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surface pencil passing through a given curve as geodesic. Li, Wang and Zhu [19] discuss the developable surface which contains a given 3D Bézier curve as geodesic and prove the corresponding conclusions in detail.

In this article we offer an alternative approach for designing developable surfaces from a given geodesic curve. In Section 2, we present, in brief, the basic definitions of differential geometry and the approach of constructing surfaces through a spatial curve. In Section 3, for a given parametric curve \( \alpha(\sigma) \), where \( \sigma \) is arc length parameter, we construct a developable surface which possesses \( \alpha(\sigma) \) as the geodesic and give the concrete expression of the resulting surface. Then, we give the conditions when the developable surface is cylinder, cone or tangent surface respectively. Also, some representative curves are chosen to construct the corresponding developable surfaces. Finally, employing some other different perspective, determination of developable surface which is also surface of revolution is given.

2 Preliminaries

The ambient space is the Euclidean space \( \mathbb{E}^3 \) and for our work we have used [1, 12] as general references. A curve is regular if it admits a tangent line at each point of the curve. In the following discussions, all curves are assumed to be regular. Given a spatial curve \( \alpha : s \rightarrow \alpha(s) \), which is parameterized by arc length parameter \( s \). We assume \( \dot{\alpha}(s) \neq 0 \) for all \( s \in [0, L] \), since this would give us a straight line. In this paper, \( \ddot{\alpha}(s) \) and \( \dddot{\alpha}(r) \) denote the derivatives of \( \alpha \) with respect to arc-length parameter \( s \) and arbitrary parameter \( r \), respectively. For each point of \( \alpha(s) \), the set \( \{ \mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s) \} \) is called the Serret–Frenet Frame along \( \alpha(s) \), where \( \mathbf{t}(s) = \dot{\alpha}(s), \mathbf{n}(s) = \dddot{\alpha}(s)/\|\dddot{\alpha}(s)\| \) and \( \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) \) are the unit tangent, principal normal, and binormal vectors of the curve at the point \( \alpha(s) \), respectively. The arc-length derivative of the Serret–Frenet frame is governed by the relations [1]:

\[
\frac{d}{ds} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},
\]

where the curvature \( \kappa(s) \) and torsion \( \tau(s) \) of the curve \( \alpha(s) \) are defined by

\[
\kappa(s) = \|\dddot{\alpha}(s)\|, \quad \tau(s) = \frac{\det(\dddot{\alpha}(s), \dddot{\alpha}(s), \dddot{\alpha}(s))}{\|\dddot{\alpha}(s)\|^2}.
\]

Although the parameter of arc length is simple for analyzing, in the majority of practical cases, the parameter of a given curve is usually not in arc length representation. We can reparameterize the given curve by using arc
length parametrization. Given the parametric curve
\[ \alpha(r) = (\alpha_1(r), \alpha_2(r), \alpha_3(r)), \ 0 \leq r \leq H, \]
where the parameter \( v \) is not the arc length. The components of the Serret–Frenet-Serret frame are defined by [1]:
\[ t(r) = \frac{\alpha'(r)}{\|\alpha'(r)\|}, \quad b(r) = \frac{\alpha'(r) \times \alpha''(r)}{\|\alpha'(r) \times \alpha''(rr)\|}, \quad n(r) = b(r) \times t(r), \quad \left( \frac{d}{dr} \right)' , \]
and the corresponding Serret–Frenet–Serret frame is given by
\[ \begin{pmatrix} t'(r) \\ n'(r) \\ b'(r) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(r) \|\alpha'(r)\| & 0 \\ -\kappa(r) \|\alpha'(r)\| & 0 & \tau(v) \|\alpha'(v)\| \\ 0 & -\tau(r) \|\alpha'(r)\| & 0 \end{pmatrix} \begin{pmatrix} t(r) \\ n(r) \\ b(r) \end{pmatrix}. \]

### 2.1 ruled surfaces

The parametrized surface
\[ P(s, u) = \alpha(s) + uL(s), \ u \in \mathbb{R}, \]
is called a ruled surface; \( \alpha(s) \) is called the base curve, and the line passing through \( \alpha(s) \) that is parallel to \( L(s) \) is called the ruling of the surface at \( \alpha(s) \). The surface \( P(s, u) \) is regular if \( P_s \times P_u \neq 0 \) for all points, where \( P_s \) and \( P_u \) are the partial derivatives of \( P(s, u) \) with respect to \( s \) and \( u \), respectively. The unit normal vector of the ruled surface \( P(s, u) \) at each point is defined by
\[ N(s, u) = \frac{P_s \times P_u}{\|P_s \times P_u\|} = \frac{\dot{\alpha}(s) \times L(s) + uL(s) \times L(s)}{\|\dot{\alpha}(s) \times L(s) + uL(s) \times L(s)\|}. \]
The base curve is not unique, since any curve of the form:
\[ c(s) = \alpha(s) - \eta(s)L(s), \]
may be used as its base curve, \( \eta(s) \) is a smooth function. If there exists a common perpendicular to two neighboring rulings on \( P(s, u) \), then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the striction curve. In (6) if
\[ \eta(s) = \frac{<\dot{\alpha}(s), L(s)>}{\|L\|^2}, \]
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then \( c(s) \) is called the striction curve on the ruled surface and it is unique. In the case of \( \eta = 0 \) the base curve is the striction curve.

The distribution parameter of \( \mathbf{P}(s, u) \) is defined by

\[
\lambda(s) = \frac{\det(\dot{\alpha}, \mathbf{L}, \dot{\mathbf{L}})}{\|\mathbf{L}\|^2}. \tag{8}
\]

The distribution parameter is well-known real integral invariants of a ruled surface and allows further classification of the ruled surface. The relation between the Gaussian curvature \( K \) and the distribution parameter is given by the formula:

\[
K(s, u) := \frac{eg - f^2}{EG - F^2} = -\frac{\lambda^2}{\lambda^2 + u^2}. \tag{9}
\]

Here \( E, F, G \) and \( e, f, g \) are the elements of the first and second fundamental forms on the surface, that

\[
E = \langle \mathbf{P}_s, \mathbf{P}_s \rangle, \quad F = \langle \mathbf{P}_s, \mathbf{P}_u \rangle, \quad G = \langle \mathbf{P}_u, \mathbf{P}_u \rangle,
\]

and

\[
e = \langle \mathbf{P}_{ss}, \mathbf{N} \rangle, \quad f = \langle \mathbf{P}_{su}, \mathbf{N} \rangle, \quad g = \langle \mathbf{P}_{uu}, \mathbf{N} \rangle,
\]

respectively. The ruled surface \( \mathbf{P}(s, u) \) is said to be developable if and if

\[
\lambda(s) = 0 \iff K(s, u) = eg - f^2 = 0. \tag{10}
\]

Geometrically this condition implies that the tangent planes along each ruling are parallel; the normal vector to the surface therefore does not depend on the parameter \( s \). Intrinsically a developable surface can be characterized as one whose Gaussian curvature is everywhere zero.

3 Developable surfaces

Developable surfaces can be briefly introduced as special cases of ruled surfaces. Intuitively, a developable surface can be obtained by twisting and bending a flat surface, such as sheet metal or paper, without stretching, compressing, or any other type of elastic deformation. Such surfaces are widely used, for example, in the manufacture of automobile body parts, airplane wings, and ship hulls.

We begin by retrieving some basic facts about developable surfaces from the viewpoint of the classical differential geometry. If the ruled surface \( \mathbf{P}(s, u) \) is a developable one, then \( \lambda \) vanishes and the we have

\[
\det(\dot{\alpha}, \mathbf{L}, \dot{\mathbf{L}}) = 0.
\]
Thus a volume formed by \( \alpha \), \( L \) and \( \dot{L} \) vanishes, i.e., they are linearly dependent. This condition is satisfied provided that there are three nonidentically vanishing functions \( \eta(s), \beta(s) \) and \( \gamma(s) \) satisfying
\[
\mu(s)\alpha + \beta(s)L + \gamma(s)\dot{L} = 0.
\] (11)

We must consider the following cases:

Case 1: \( \mu = 0 \)

Since \( \langle L, \dot{L}(s) \rangle = 0 \) it follows immediately Eq. (11) is only satisfied when \( L \) is a constant vector, i.e., \( P(s, u) \) is a part of a cylinder.

Case 2: \( \mu \neq 0 \) from Eq. (11) it follows:
\[
\alpha = \zeta(s)L + v(s)\dot{L},
\] (12)

where
\[
\zeta(s) = -\frac{\beta}{\mu}, \quad v(s) = -\frac{\gamma}{\mu}.
\]

Differentiating Eq. (6) and using (12), we get
\[
\dot{c}(s) = (\zeta(s) - \dot{\eta}(s))L(s) + (v(s) - \eta(s))\dot{L}.
\] (13)

The condition for \( c \) to be striction curve is equivalent to \( \dot{c} \) and \( \dot{L} \) are perpendicular to each other. Therefore, we may conclude that the ruling is parallel to the first derivative of the striction curve, which is also the tangent of the striction curve, i.e.
\[
\dot{c}(s) = (\zeta(s) - \dot{\eta}(s))L(s).
\] (14)

We must hence consider the following sub-cases:

Case a: \( \zeta(s) = \dot{\eta}(s) \)

In this case (14) yields \( c = c_0 \) is a constant vector. So, \( P(s, u) \) is a part of a cone.
\[
P(s, u) = c_0 + (\eta(s) + u)L(s), \quad u \in \mathbb{R}.
\] (15)

Case b: \( \zeta(s) \neq \dot{\eta}(s) \)

In this case from (14) it follows
\[
L(s) = \frac{\|\dot{c}(s)\|\dot{c}(s)}{(\zeta(s) - \dot{\eta}(s))\|c(s)\|}.
\]

Therefore from (6)
\[
\alpha(s) = c(s) + \frac{\eta(s)}{(\zeta(s) - \dot{\eta}(s))}\|c(s)\|\frac{\dot{c}(s)}{\|c(s)\|},
\]
i.e., \( P(s, u) \) is a part of the a tangential developable of \( c(s) \)
\[
P(s, u) = c(s) + (u + \eta(s))\left(\frac{\|\dot{c}(s)\|}{(\zeta(s) - \dot{\eta}(s))}\right)\frac{\dot{c}(s)}{\|c(s)\|}.
\] (16)
3.1 Developable surface with a common geodesic

Now, we illustrate an approach to developable surface design, by designing some developable surfaces based on some of the previous choices of the base curve. For this purpose, without loss of generality, let $L(\sigma)$ is strictly connected to the Serret–Frenet frame of a given spatial curve $\alpha(\sigma)$, parameterized by arc length parameter $\sigma$ in the form:

$$L(\sigma) = \ell_1(\sigma) t(\sigma) + \ell_2(\sigma) n(\sigma) + \ell_3(\sigma) b(\sigma),$$  \hspace{1cm} (17)

where $\ell_i = \ell_i(s)$ (i=1, 2, 3) are scalar functions of the arc length parameter of the prescribed curve $\alpha = \alpha(s)$. If $L$ moves along $\alpha = \alpha(s)$, the corresponding ruled surface is:

$$P(s, u) = \alpha(\sigma) + u L(s), \quad u \in \mathbb{R} \ \bigg\{ \begin{array}{l} \ell_1^2 + \ell_2^2 + \ell_3^2 = 1, \quad L(s) \neq 0. \end{array} \bigg\}$$  \hspace{1cm} (18)

To solve our constrained design problem, rather than employing the metric definition of a geodesic in terms of length minimization, we resort to the alternative characterization using local properties. A geodesic $\alpha(\sigma)$ on a surface is a curve whose osculating plane, i.e. the plane generated by the tangent $t$ and normal $n$ to the curve, contains the normal vector $N$ to the surface. Namely, the vectors $n$ and $N$ are always parallel, or the binormal $b$ to the curve is tangent to the surface. Thereby, from Eqs. (1), (5) and (18), the unit normal vector $N$ at the point $(\sigma, 0)$ is

$$N(s, 0) = \frac{\alpha(s) \times L(s)}{\|\alpha(s) \times L(s)\|} = \frac{\ell_2 b - \ell_3 n}{\sqrt{\ell_2^2 + \ell_3^2}}$$  \hspace{1cm} (19)

Thus the base curve $\alpha(s)$ is a geodesic curve, i.e. $N(s, 0) \| n(s)$ if and only if $\ell_2 = 0$ and $\ell_3 \neq 0$. It follows from the above that

$$P(s, u) = \alpha(\sigma) + u L(s), \quad L(s) = \ell_1 t(s) + \ell_3 b(s).$$  \hspace{1cm} (20)

It can be immediately seen from Eqs. (8) and (20) that $P(s, u)$ is developable if and only if

$$\lambda = 0 \Leftrightarrow (\kappa \ell_1 - \tau \ell_3) \ell_3 = 0.$$  \hspace{1cm} (21)

We will now investigate this condition in detail: if $\ell_3 = 0$, then $\alpha(s)$ can not be geodesic (In fact we have imposed $\ell_3 \neq 0$). Hence, according to the assumption of $\alpha(s)$ being a geodesic, we have

$$\kappa \ell_1 - \tau \ell_3 = 0.$$  \hspace{1cm} (21)

By substitution:

$$\ell_1(s) = \langle L, t \rangle = \cos \varphi, \quad \ell_3(s) = \langle L, b \rangle = \sin \varphi \neq 0,$$  \hspace{1cm} (22)
from (21) we have:

\[ \frac{\tau}{k}(s) = \cot \varphi(s). \]

As a result the following corollary can be given:

**Lemma 1.** Planar curves and helices are geodesic curves on developable surfaces.

Together with the above conditions, we then have the following theorem:

**Theorem 1.** The necessary and sufficient condition for \( P(s, u) \) being a developable surface with \( \alpha(s) \) as a geodesic is that there exist a parameter \( u \in \mathbb{R} \) and a function \( \varphi(s) \), so that \( P(s, u) \) can be represented by

\[
P(s, u) = \alpha(s) + uL(s), \quad L(s) = \cos \varphi t(s) + \sin \varphi b(s), \ \varphi \neq 0. \tag{23}
\]

By Theorem 1, we not only prove the existence of the developable surface, but also give the concrete expression of the surface. The developable surfaces are not unique. When we choose different \( \varphi_0 \), we can get different surfaces. We can control the shape of the surfaces by the value of \( \varphi(s) \). According to the three types of developable surface we give the conditions when the surface (23) is a cylinder, cone or tangent surface, respectively.

Hence, the following corollaries discuss the relationship between the given geodesic and its associated developable surface.

**Corollary 1.** The developable surface \( P(s, u) \) is a cylinder surface if and only if \( \alpha(s) \) is a planar curve or is a circular helix.

**Proof:** Obviously, the conclusion is hold if the curve \( \alpha(s) \) is a planar curve. \( P(s, u) \) is a cylindrical surface if and only if

\[ L \times \dot{L} = 0 \iff \dot{\varphi} n = 0 \iff \varphi = \text{const}. \]

This means that

\[ \frac{\tau}{\kappa} = \text{const}. \]

The ratio of curvature to torsion is a constant, and so the curve is a circular helix. The conclusion holds

**Corollary 2.** The ruled surface \( P(s, u) \) is a circular cone if and only if \( \alpha(s) \) is a generalized helix satisfying the following condition:

\[ \frac{\tau}{\kappa} = \frac{s}{c}. \]

**Proof:** As mentioned above, from Eqs. (6) and (20), we have

\[ \dot{c}(s) = (1 - \eta \cos \varphi + \eta \dot{\varphi} \sin \varphi)t - (\eta \sin \varphi + \eta \dot{\varphi} \cos \varphi)b, \]

which follows that \( P(s, u) \) is a circular cone if and only if

\[ 0 = (1 - \eta \cos \varphi + \eta \dot{\varphi} \sin \varphi)t - (\eta \sin \varphi + \eta \dot{\varphi} \cos \varphi)b. \]
We have found, by equating the coefficients of \( t \) and \( b \), that:

\[
\begin{align*}
1 - \dot{\eta} \cos \varphi + \eta \dot{\varphi} \sin \varphi &= 0 \iff 1 - \frac{d}{ds}(\eta \cos \varphi) = 0 \iff \eta \cos \varphi = s, \\
\dot{\eta} \sin \varphi + \eta \dot{\varphi} \cos \varphi &= 0 \iff \frac{d}{ds}(\eta \sin \varphi) = 0 \iff \eta \sin \varphi = c.
\end{align*}
\]

Hence, we have

\[
\frac{\tau}{\kappa} = \frac{s}{c},
\]

which ends the proof.

**Corollary 3.** The ruled surface \( P(s, u) \) is a tangential developable if and only if \( \alpha(s) \) does not satisfy the Corollaries (1) and (2).

**Proof:** According to the proof of Corollary 2, when \( \frac{\tau}{\kappa} \neq \frac{s}{c} \), we have \( \dot{c}(s) \neq 0 \).

Since \( \det(\dot{c}, \dot{L}, \ddot{L}) = 0 \), \( \langle \dot{c}, \dot{L} \rangle = 0 \) and \( \langle L, \dot{L} \rangle = 0 \), we can get \( \dot{c} \parallel L \). This means the surface (23) is a tangent surface. The conclusion holds.

According to Corollary 3, the tangent developable surface can be parameterized by \( P(s, u) = \alpha(s) + u\dot{\alpha}(s) \), the curve \( \alpha(s) \) is called the curve of regression.

### 3.2 Examples

Now, we are going to deals with construct some representative examples to verify the method.

**Example 1:** In this example, we construct a geodesic surface pencil in which all the surfaces share a geodesic circular helix represented as:

\[
\alpha(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \frac{s}{c} \right), \quad a > 0, \ b \neq 0, \ c = \sqrt{a^2 + b^2}, \ -\sqrt{5}\pi \leq s \leq \sqrt{5}\pi.
\]

The Serret–Frenet frame is

\[
\begin{align*}
t(s) &= (-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, b \frac{s}{c}), \\
n(s) &= (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0), \\
b(s) &= (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{s}{c}).
\end{align*}
\]

It is easy to show that

\[
\kappa = \frac{a}{a^2 + b^2} > 0, \quad \tau = \frac{b}{a^2 + b^2} \neq 0, \quad \frac{\tau}{\kappa} = \frac{b}{a}.
\]

Now let

\[
\cos \varphi = \frac{b}{\sqrt{a^2 + b^2}}, \quad \sin \varphi = \frac{a}{\sqrt{a^2 + b^2}}.
\]

According to Corollary 1, the cylinder surface pencil can be expressed as

\[
P(s, u) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \frac{s}{c} \right) + u \left( \cos \varphi, 0, \sin \varphi \right) \left( \begin{array}{ccc}
-\frac{a}{c} & \frac{b}{c} & \frac{a}{c} \\
-\cos \frac{s}{c} & -\sin \frac{s}{c} & 0 \\
\frac{b}{c} & -\frac{b}{c} \cos \frac{s}{c} & \frac{a}{c}
\end{array} \right).
\]
If we take $a = 2$ and $b = 1$, then we immediately obtain a member of this family as:

$$P(s, u) = \left( 2 \cos \frac{s}{\sqrt{5}}, 2 \sin \frac{s}{\sqrt{5}}, \frac{s}{\sqrt{5}} \right) + u \left( \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right) \left( \begin{array}{ccc} \frac{2}{\sqrt{5}} \sin \frac{s}{\sqrt{5}} & \frac{2}{\sqrt{5}} \cos \frac{s}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\cos \frac{s}{\sqrt{5}} & -\sin \frac{s}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} \sin \frac{s}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \cos \frac{s}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{array} \right).$$

This surface is shown in Fig. 1, in which $s \in [-\sqrt{5} \pi, \sqrt{5} \pi]$ and $u \in [0, 5]$.

**Example 2:** If $p_0 = (0, 0, 0)$, $p_1 = (0, 1, 1)$ and $p_2 = (1, 2, 0)$ are points in the Euclidean 3-space $E^3$, then the quadratic Bézier curve can be expressed as

$$\alpha(r) = Z_0(r)p_0 + Z_1(r)p_1 + Z_2(r)p_2, \ 0 \leq r \leq 1.$$ 

where

$$Z_0(r) = (1 - r)^2, \ Z_1(r) = 2r(1 - r), \ Z_2(r) = r^2,$$

are the blending functions of the curve $\alpha(r)$. It is easy to show that

$$\kappa(r) = \frac{1}{2} \sqrt{\frac{6}{5r^2 - 4r + 2}}, \ \tau(r) = 0.$$ 

After simple computation, we get

$$t(r) = \frac{(r, 1, -2r + 1)}{\sqrt{5r^2 - 4r + 2}}, \ n(r) = \frac{(-2r, 2 - 3r, -(2r + r))}{\sqrt{6(5r^2 - 4r + 2)}}, \ b(r) = \frac{(-2, 1, 1)}{\sqrt{6} \sqrt{6} \sqrt{6}}.$$ 

According to Corollary 1, the developable surface $P(r, u)$ is a cylinder surface. Hence, we obtain the cylinder surface as

$$P(r, u) = \alpha(r) + ub(r) = \left( r^2 - \frac{u}{\sqrt{6}}, 2r + \frac{u}{\sqrt{6}}, 2r - 2r^2 - \frac{u}{\sqrt{6}} \right).$$
This surface is shown in Fig. 2, in which $u \in [-1, 1]$.

**Example 3.** In this example, we construct a developable spline surface which possesses a quadratic spline curve as a geodesic. Let the knot sequence be $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the corresponding basis functions of quadratic B-spline are

$$N_{i,2}(r) = \begin{cases} \frac{(r-t_i)(t_{i+2}-r)}{2}, & [t_i, t_{i+1}] \\ \frac{(r-t_{i+1})(t_{i+3}-r)}{2}, & [t_{i+1}, t_{i+2}], \\ \frac{(r-t_{i+3})^2}{2}, & [t_{i+2}, t_{i+3}]. \end{cases}$$

Given the control points $P_0 = (0, 0, 0), P_1 = (1, 2, 0), P_2 = (3, 3, 0), P_3 = (4, 2, 0), P_4 = (5, 3, 0), P_5 = (6, 3, 0), P_6 = (7, 2, 0), P_7 = (8, 4, 0)$, the spline curve is

$$\alpha(r) = \sum_{i=0}^{7} P_i N_{i,2}(r), \quad r \in [2, 8],$$

which consists of six quadratic curve segments. We use $\alpha_i(r), (i = 1, \ldots, 6)$ represent these quadratic segments. For the segment

$$\alpha_1(r) = \left(\frac{r^2}{2} - r + \frac{1}{2}, -\frac{r^2}{2} + 4r - 5, 0\right), \quad r \in [2, 3]$$

By simple computation, we have $\tau_1 = 0$, then according to the Eq. (23), the cylinder surface is

$$P_1(r, u) = \left(\frac{r^2}{2} - r + \frac{1}{2}, -\frac{r^2}{2} + 4r - 5, u\right).$$
In the same way, for the segments
\[
\alpha_2(r) = (-r^2 + 5r - \frac{17}{2}, -r^2 + 7r - \frac{19}{2}, 0), \quad \tau_2 = 0,
\]
\[
\alpha_3(r) = (r - \frac{1}{2}, r^2 - 9r + \frac{45}{2}, 0), \quad \tau_3 = 0.
\]
\[
\alpha_4(r) = (r - \frac{1}{2}, -r^2 + 6r - 15, 0), \quad \tau_4 = 0,
\]
\[
\alpha_5(r) = (r - \frac{1}{2}, -r^2 + 6r - 15, 0), \quad \tau_5 = 0,
\]
\[
\alpha_6(r) = (r - \frac{1}{2}, \frac{3}{2}r^2 - 22r + 83, 0), \quad \tau_6 = 0.
\]

According to Corollary 1, the cylinder surface pencil can be expressed as
\[
\begin{align*}
P_2(r, u) &= (-r^2 + 5r - \frac{17}{2}, -r^2 + 7r - \frac{19}{2}, u), \quad r \in [3, 4], \\
P_3(r, u) &= (r - \frac{1}{2}, r^2 - 9r + \frac{45}{2}, u), \quad r \in [4, 5], \\
P_4(r, u) &= (r - \frac{1}{2}, -r^2 + 6r - 15, u), \quad r \in [5, 6], \\
P_5(r, u) &= (r - \frac{1}{2}, -r^2 + 6r - 15, u), \quad r \in [6, 7], \\
P_6(r, u) &= (r - \frac{1}{2}, \frac{3}{2}r^2 - 22r + 83, u). \quad r \in [7, 8].
\end{align*}
\]

Finally, then the hole cylinder surface pencil is shown in Fig. 3.

Fig. 3.

**Example 4.** (Developable surfaces from surfaces of revolution): We select a parametric curve
\[
\Gamma(u) = (x(u), 0, z(u)), \quad u \in [u_0, u_1].
\]

Obviously, it is in the oXZ plane. Let it satisfy the following conditions:
\[
\Gamma(u_0) = (a, 0, 0), \quad \Gamma'(u_0) = (0, 0, 1), \quad a > 0.
\]  

(24)
We revolve the curve $\Gamma(u)$ around the $z$-axis, then obtain a surface of revolution (see Fig. 4):

$$\tilde{P}(r, u) = (x(u) \cos r, x(u) \sin r, z(u)), \quad (u, r) \in [u_0, u_1] \times [0, 2\pi]. \quad (25)$$

Suppose now we are given a circle

$$\alpha(r) = (a \cos r, a \sin r, 0), \quad a > 0, \quad 0 \leq r \leq 2\pi.$$ 

It is easy to show that

$$t(r) = (-\sin r, \cos r, 0), \quad n(r) = (-\cos r, -\sin r, 0), \quad b(u) = 0(0, 0, 1),$$

and $\|\alpha'(u)\| = a, \kappa = 1/a, \tau = 0$, then $\varphi = \pi/2$. Obviously, it satisfies Theorem 1. Thus the developable surface pencil can be represented as

$$P(r; u; a) = (a \cos r, a \sin r, 0) + u(0, 0, 1)$$

$$\begin{pmatrix} -\sin r & \cos r & 0 \\ -\cos r & -\sin r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Hence, we have $P(r, u; a) = \tilde{P}(r, u)$ if and only if

$$z(u) = u, \quad x(u) = \sqrt{a^2 + u^2}$$

Thereby, the parametric curve $\Gamma(u)$ is presented by

$$\Gamma(u) = (\sqrt{a^2 + u^2}, 0, u), \quad u \in [u_0, u_1],$$

Note that it satisfies the conditions in Eq. (24). So, the developable surface pencil, from surfaces of revolution with a geodesic circle, is expressed as

$$P(r; u; a) = \left(\sqrt{a^2 + u^2} \cos r, \sqrt{a^2 + u^2} \sin r, u\right), \quad (u, r) \in [u_0, u_1] \times [0, 2\pi].$$

So, if we choose $u \in [-2, 2], \quad r \in [0, 2\pi]$, then for $a = 2$, the corresponding developable surface is shown in Fig. 5.
Example 5: In this Example, surfaces of revolution which are developable surfaces are constructed. For this purpose, from Eq. (25), the unit normal vector at the point $P(r, u)$ is

$$N(r, u) = \frac{P_r \times P_u}{\|P_r \times P_u\|} = \frac{1}{D} (-z'(u) \cos r, -z'(u) \sin r, x'(u)),$$

where $D = \sqrt{x'^2 + z'^2}$. The elements of second fundamental form are given by

$$e = \frac{-z' x'' + x' z''}{D}, \quad f = 0, \quad g = \frac{x' z'}{D}.$$

As we mentioned in section 2, $P(r, u)$ is developable if and only if its Gaussian curvature is everywhere zero, so, we must have

$$K = 0 \iff e g - f^2 = 0 \iff e g = 0 \iff g = 0 \text{ or } e = 0.$$

In the case of $g = 0$, and $e \neq 0$, we get

$$z' = 0 \Rightarrow z = c_1 (\text{real const.}).$$

Hence, under these conditions, by rotating we obtain a part of plane. The plane is of course an excellent example of developable surface. In this case the problem does not have meaning. Therefore, suppose $z'$ is non zero at some point, meaning $z' \neq 0$ in the interval $[u_0, u_1]$. Then, in this interval, we may assume the curve parametrized as

$$\alpha(u) = (x(u), 0, u), \quad u \in [u_0, u_1].$$
Then
\[ x'' = 0 \Rightarrow x = cu + a. \]

Therefore, the corresponding surfaces of revolution which are developable surfaces have the form

\[ P(r, u; a, c) = ((cu + a) \cos r, (cu + a) \sin r, u), \quad (u, r) \in [u_0, u_1] \times [0, 2\pi], \]

where \( a > 0 \) and \( c \) are two arbitrary real number constants. If we choose \( a = 1, c = -3, r \in [0, 2\pi] \), the surface is shown in Fig 6. Fig 7. shows the surface with \( a = 1, c = 0 \).

4 Conclusion

In this study, we have presented an approach to design a developable surface using the surface pencil passing through a given curve. By representing the surface by the combination of the given curve, and the three vectors decomposed along the directions of Serret–Frenet frame, we derive the necessary and sufficient conditions for a surface to be developable. In addition, we have studied the problem of requiring the given curve to be a geodesic. Research results show that the given curve can be classified into three kinds. They are, respectively, isogeodesic to different types of developable surface, that is, cylinders, cones and tangent surfaces. In addition, we have solved the problem of requiring the surfaces that are revolution surfaces and at the same time developable. Meanwhile, examples illustrates the application of the obtained formulae are introduced.
References


An approach for designing a developable surface


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