Computing Generators of Second Homotopy Module Using Tietze Transformation Methods

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Abstract

This paper discusses the relationship of second homotopy module for two different presentations defining a similar group. These two presentations can be transformed to each other using Tietze transformation. This relationship was determined by considering the generators of second homotopy module for both presentations.

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1 Introduction

Let \( P = \langle x; r \rangle \) be a presentation for a group \( G \). Then we have the first fundamental group \( \pi_1(P) \) over \( P = \langle x; r \rangle \). The elements of \( \pi_1(P) \) are equivalent
classes of words [W]. Moreover, we can have a picture $\mathbb{P}$ over $\mathcal{P}$. A picture $\mathbb{P}$ over $\mathcal{P}$ is an object consist of disjoint arcs labelled by element of $x$, discs labelled by element of $r$, and a boundary disc with a basepoint.

A picture $\mathbb{P}$ over $\mathcal{P}$ is a spherical picture if all arcs in $\mathcal{P}$ do not touch the boundary disc. Then we have the second homotopy module $\pi_2(\mathcal{P})$. The elements of $\pi_2(\mathcal{P})$ are equivalent classes of spherical picture $[\mathbb{P}]$.

Let a group $G$ defined by two group presentation, say $\mathcal{P}_1$ and $\mathcal{P}_2$. There are some alternations one can make to presentation $\mathcal{P}_2$ which result in presentation of a group isomorphic to the original $\mathcal{P}_1$ (see [1] and [5]). These are called Tietze transformations. Tietze transformation are simply the obvious ways of transforming a finite presentation $\langle x : r \rangle$.

Tietze transformation are useful in special cases for showing that two given presentations define isomorphic group, and, in particular, for simplifying a given presentation. We describe this transformations as follows.

Let $\mathcal{P}_1 = \langle x : r \rangle$ dan $\mathcal{P}_2 = \langle y : s \rangle$ be two presentations of the group $G$. Then there are the following Tietze transformations which may be performed upon the group presentations:

(T1) If the word $S$ is derivable from $\{r\}$, then add $S$ to the list of relators.
(T2) If the word $S$ is derivable from $\{r\}\setminus S$, remove $S$ from the list relators.
(T3) If $R$ is word in the $x$, and $y$ is some symbol not in the generating set, add $y$ to the generating set and add word $y^{-1}y$ to the relator set.
(T4) If there is a relator of the form $y^{-1}\mathbb{P}$, $y$ not appearing in $R$, delete this relator and delete $y$ from the generating set, replacing all order occurrences of $y$ in the relator words with $R$.

The problem of $\pi_2(\mathcal{P})$ is to compute its generator (see [4]). Suppose that $\mathbb{P}$ is set of spherical pictures over $\mathcal{P}$. If all spherical pictures $\mathbb{P}$ are equivalent to the empty picture (relative to $\mathcal{P}$) then we say that $\mathbb{P}$ generates $\pi_2(\mathcal{P})$. In this paper we are going to determine the relationship between generators of $\pi_2(\mathcal{P}_1)$ and $\pi_2(\mathcal{P}_2)$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ define the same group.

We are going to prove:

**Theorem 1.** Let $\mathcal{P}_1 = \langle x : r \rangle$ and $\mathcal{P}_2 = \langle x : r, T \rangle$ be a presentation define a group $G$, where $T$ is a cyclically reduced word define and $T \sim 1$ (relative to $r$). If $\pi_2(\mathcal{P}_1)$ is generated by $\mathbb{P}_1$ then $\pi_2(\mathcal{P}_2)$ is generated by $\mathbb{P}_1 \cup \{[\mathbb{P}_T]\}$, where $\mathbb{P}_T$ is spherical picture having a T-disc joining to a picture $\mathbb{P}$ over $\mathcal{P}_1$.

**Theorem 2.** Let $\mathcal{P}_1 = \langle x : r \rangle$ and $\mathcal{P}_2 = \langle x, y : r, y = S \rangle$ be a presentation define a group $G$, where $S$ a word on $x$. Then $\pi_2(\mathcal{P}_1)$ has same generator with $\pi_2(\mathcal{P}_2)$.

Proof of these theorem by using operations on picture and van Kampen’s Lemma and will be given on section 3.
2. Picture and Operation on Picture

A picture $\mathcal{P}$ in $\mathcal{P} = \langle x: r \rangle$ is an object consist of disjoint arcs labeled by element of $x$, discs labeled by element of $r$ and a boundary disc with a basepoint (see [4] and [2]). A picture $\mathcal{P}$ in $\mathcal{P} = \langle x: r \rangle$ is a spherical picture if all arcs in $\mathcal{P}$ do not touch the boundary disc. Certain basic operation can be applied to a picture (spherical picture) $\mathcal{P}$ as follows: deletion and insertion floating circle, deletion and insertion floating semicircle, deletion and insertion folding pair and bridge move (see [3]), as depicts below.

Two spherical pictures $\mathcal{P}_1$ and $\mathcal{P}_2$ are said to be equivalent if either: (a) both are spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion folding pair and bridge move; or (b) both are not spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion semicircle, deletion and insertion folding pair and bridge move.

The equivalent class containing the spherical picture $\mathcal{P}$ is denoted by $[\mathcal{P}]$. The equivalent class containing the empty picture (null) is denoted by $\mathcal{P}$. The addition $\mathcal{P}_1 + \mathcal{P}_2$ is defined by drawing $\mathcal{P}_1$ and $\mathcal{P}_2$.

Set of equivalent classes of spherical picture with binary operation $[\mathcal{P}_1] + [\mathcal{P}_2] = [\mathcal{P}_1 + \mathcal{P}_2]$ form a abelian group under this operation and this abelian group is right $\mathbb{Z}G$-module, where the action is given by $[\mathcal{P}]\overrightarrow{W} = [\mathcal{P}\overrightarrow{W}]$ ($\overrightarrow{W}$ denotes the element of $G$ represented by $W$). This module is called the second homotopy module of $\mathcal{P}$, denoted by $\pi_2(\mathcal{P})$.

A set $\mathcal{P}$ of spherical pictures over $\mathcal{P}$ will be called a generating set of pictures if $\{[\mathcal{P}]: \mathcal{P} \in \mathcal{P}\}$ generates the $\mathbb{Z}G$-module $\pi_2(\mathcal{P})$ (see [6]). It follow [4], that $\mathcal{P}$ is generating set if and only if every spherical picture over $\mathcal{P}$ can be transformed to empty picture by operations: bridge moves, insertion/deletion of floating circles, insertion/deletion of folding pairs, insertion/deletion of pictures from $\pm \mathcal{P}$.

Consider a collection $\mathcal{S}$ of spherical pictures. Now, we define two extended operation on pictures as follows:

1). (Deletion of an $\mathcal{S}$-picture) If there is a simple closed path in a picture such that
the part of the picture enclosed by the simple closed path is a copy of a spherical picture.

2). (Insertion of an \( S \)-picture) The opposite of 1).

Two pictures will be said to be equivalent (relative \( S \)) if either: a). the pictures are both spherical and one can be transformed to the other by a finite number of operation deletion and insertion floating circle, deletion and insertion folding pair, bridge move, and deletion and insertion \( S \)-picture; or b). the picture are not both spherical and one can be transformed to the other by a finite number of operations deletion and insertion floating circle, deletion and insertion floating semicircle, deletion and insertion folding pair, bridge move and deletion and insertion \( S \)-picture (see [3]).

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1

Suppose that \( P_1 = \langle x: r \rangle \) is generated by \( P_1 \). Consider that:

\[
P_1 = \langle x: r \rangle \xrightarrow{T_1} P_2 = \langle x: r, T \rangle
\]

is a one of operation Tietze transformation. From (\( ^{(*)} \)) we know that \( T \) is a relator which is add on \( P_2 \) and \( T \sim r \). Based on van Kampen Lemma, there is a picture \( \mathcal{Q} \) over \( P_1 \) where \( W(\mathcal{Q}) = T \). Then picture

![Figure 1. Spherical picture \( W(\mathcal{Q}) = T \)](image)

is a spherical picture.

Since \( \mathcal{Q} \) has \( T \)-disc, then it could not be got \( \mathcal{Q} \) of picture in \( P_1 \). Therefore, \( \mathcal{Q} \) is one of generator of \( P_2 \). From this, we have generator of \( P_2 \) is generator of \( P_1 \) and picture \( \mathcal{Q} \).

Let \( \mathbb{P} \) spherical picture in \( P_2 \). We consider two case, i.e. 1). \( \mathbb{P} \) has no \( T \)-disc, and 2). \( \mathbb{P} \) has \( T \)-disc.

If \( \mathbb{P} \) has no \( T \)-disc, then \( \mathbb{P} \) is picture in \( P_1 \). So \( P_1 \sim 1 \) (relative \( P_1 \)). If \( \mathbb{P} \) has \( T \)-disc,
then we may put the picture on Figure 1. on left side Figure 2. We apply bridge move operation to delete the inverses pair $T$-disc. The operation is applied until there is no $T$-disc in $P$. So we deduce that $\pi_2(P_2)$ is generated by $P \cup \{[P_T]\}$, where $P_T$ is spherical picture having a $T$-disc joining to a picture $P_T$ over $P_1$. 

**Proof of Theorem 1.2**

Suppose that $P_1 = \langle x: r \rangle$ is generated by $P$. Consider that

$$P_1 = \langle x: r \rangle \xrightarrow{T3} P_2 = \langle x, y: r, y = S \rangle$$

is one of Tietze transformation operations. Recall that if $P_1 = \langle x: r \rangle$ with generator $P$ is spherical picture with labeled $r$. By using (T3) operation is added a new generator in $P_1$, say $y$, where $y$ is labeled by $S$, so we have a new presentation, that is $P_2 = \langle x, y: r, y = S \rangle$.

Suppose that $Q$ is generator of $\pi_2(P_2)$, but it isn’t generator of $\pi_2(P_1)$. So $Q$ must have disc $yS^{-1}$. Since spherical picture arc $y$ is related to a disc which is inverses pair, so we can use bridge move operation. We use this operation until there are no disc $S$. Therefore, generator of $P_2$ is labeled by $r$, thus we have generator of $\pi_2(P_2)$ is $P$.

**Corrolari 1.** Let $P_1 = \langle x: r, T \rangle$ and $P_2 = \langle x: r \rangle$ be a presentation define a group $G$, where $T$ is a cyclically reduced word define and $T \sim_r 1$ (relative to $r$). Let $\pi_2(P_1)$ is generated by $P$ then $\pi_2(P_2)$ is generated by all disc are labeled by $T$ changed with a picture in $\langle x: r \rangle$ is labeled $T$.

**Corrolari 2.** Let $P_1 = \langle x, y: r, y = S \rangle$ and $P_2 = \langle x: r \rangle$ be a presentation define a group $G$, where $S$ a word on $x$. Let $\pi_2(P_1)$ is generated by $P$ then $\pi_2(P_2)$ is generated by same pictures in $P$ with arc $y$ changed by arc $S$. 

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Figure 2. Spherical Picture $P$ has $T$-disc and picture $S$ has no $T$-disc.
References


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