Existence Conditions for Finite Reduction of Knowledge

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Abstract

In rough set theory, knowledge reduction is one of important research topics, and also a critical step of knowledge acquisition. This paper studies knowledge bases on infinite universe by considering the problem of existence of finite reductions of knowledge in an infinite knowledge base. Some sufficient and/or necessary conditions for existence of finite reductions of a knowledge base are given. Some examples are constructed to reveal various cases of existence of knowledge reductions.

Keywords: infinite universe; knowledge base; finite reduction; existence condition

1 Introduction

Rough set theory [6, 8] is an important tool for dealing with fuzzyness and uncertainty of knowledge. Basic opinion in rough set theory is that the

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knowledge (human intelligence) is the ability to classify elements [3, 5, 8]. Abstractly speaking, knowledge is a family of classification patterns in some interesting fields, providing us some facts from which one can deduce new facts [4, 7]. Given a universe $U$ and a family of equivalent relations on $U$, the pair $K = (U, \mathcal{P})$ is called a knowledge base. Generally, it is well-known that elements in a knowledge base is not of the same importance, some even are redundant. So we often consider reductions of a knowledge base by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification. In rough set theory, knowledge reduction is one of important research topics, and also a critical step of knowledge acquisition.

In classic rough set theory, the universe one deals with normally is a nonempty finite set. In this case a knowledge base is finite and reductions always exist. For infinite universe, this is not the case. This paper will study knowledge bases on infinite universe by considering the problem of existence of finite reductions of knowledge in an infinite knowledge base. Some sufficient and/or necessary conditions for existence of finite reductions of a knowledge base are given. Some examples are constructed to reveal various cases of existence of knowledge reductions.

2 Preliminaries

We give some basic concepts and results which will be used in the sequel. Most of them come from [2, 8]. For other unstated concepts please refer to [1, 6].

**Definition 2.1.** Let $U$ be a nonempty set, $\mathcal{C} \subseteq 2^U$. If the following conditions are fulfilled,

1. for arbitrary subsets $A_1, A_2 \in \mathcal{C}$, if $A_1 \neq A_2$, then $A_1 \cap A_2 = \emptyset$;
2. $\cup_{A_i \in \mathcal{C}} A_i = U$,

then $\mathcal{C}$ is called a partition of $U$.

**Definition 2.2.** Let $U \neq \emptyset$ be a set and $R$ an equivalent relation on $U$. Set for $a \in U$, $[a]_R = \{b \in U \mid aRb\}$, then $[a]_R$ is called the equivalence class of $a$, shortly denoted by $[a]$. Set $U/R = \{[a] \mid a \in U\}$, then $U/R$ is called the quotient set, or the partition of $U$ w.r.t $R$.

**Remark 2.3.** Given a partition $\mathcal{C}$ on $U$, then $\mathcal{C}$ uniquely determines an equivalent relation $R$ such that $U/R$ is exactly $\mathcal{C}$. 
Definition 2.4. (1) Let $U \neq \emptyset$ be a set and $\mathcal{P} \neq \emptyset$ a family of equivalent relations on $U$. Then the pair $K = (U, \mathcal{P})$ is called a knowledge base, and $U$ is called the universe of $K$. Set $\text{ind}(\mathcal{P}) = \cap_{R \in \mathcal{P}} R$, then $\text{ind}(\mathcal{P})$ is still an equivalent relation on $U$, and is called the indiscernible relation of $\mathcal{P}$.

(2) Let $R \in \mathcal{P}$, then $R$ is said to be not necessary if $\text{ind}(\mathcal{P}) = \text{ind}(\mathcal{P} - \{R\})$. Otherwise, $R$ is said to be necessary. We say that $\mathcal{P}$ is independent if every element in $\mathcal{P}$ is necessary.

Definition 2.5. Let $K = (U, \mathcal{P})$ be a knowledge base, $Q \subseteq \mathcal{P}$. We say $Q$ a reduction of $\mathcal{P}$ if $\text{ind}(Q) = \text{ind}(\mathcal{P})$ and $\forall R \in Q$, $\text{ind}(Q) \neq \text{ind}(Q - \{R\})$. In this case, we also say that $Q$ is a reduction of $K$.

Definition 2.6. For a knowledge base $K = (U, \mathcal{P})$ on $U$, if $\mathcal{P}$ has only finite equivalent relations, then $K$ is called a finite knowledge base; if $\mathcal{P}_0 \subseteq \mathcal{P}$ is a reduction of $K$ and has only finite elements, then $\mathcal{P}_0$ is called a finite reduction of $K$.

Lemma 2.7. (see [8, Theorem 1.9]) A (finite) reduction always exists for a knowledge base on a finite universe.

Definition 2.8. Let $L$ be a poset.

(1) if $\forall a, b \in L$, $a \leq b$ or $b \leq a$ holds, then “$\leq$” is said a linear order, and $L$ is called a total-ordered set or a chain.

(2) $L$ is called an anti-chain if $\forall a, b \in L$, neither $a \leq b$ nor $b \leq a$ holds.

Definition 2.9. Let $L$ be a poset. If every pair of elements in $L$ has a meet, then $L$ is called an inf-semilattice or briefly semilattice. A non-empty subset $X \subseteq L$ is said to be filtered if every pair of elements $a, b \in X$, there is $c \in X$ such that $c \leq a$ and $c \leq b$.

Remark 2.10. Every semilattice itself is filtered.

3 Conditions for existence of finite reductions

This section will give existence conditions of finite reductions of infinite knowledge bases. Firstly, by Definition 2.6 and Lemma 2.7, we immediately have the following remark.

Remark 3.1. Every finite knowledge base on an infinite universe has (finite) reductions.
Definition 3.2. If \((\mathcal{P}, \subseteq)\) is a chain (resp., an anti-chain, a filtered set, a semilattice), then \(K = (U, \mathcal{P})\) is said to be a chain (resp., an anti-chain, a filtered set, a semilattice).

Remark 3.3. Let \(K = (U, \mathcal{P})\) be a knowledge base. If in the set inclusion order, the poset \((\mathcal{P}, \subseteq)\) has the least element, then \(K\) has a finite reduction.

Proof. Let \(R \in \mathcal{P}\) be the least element in the poset \((\mathcal{P}, \subseteq)\). Then it is easy to check that \(\{R\}\) is a finite reduction of \(K\). \(\Box\)

Theorem 3.4. Let \(K = (U, \mathcal{P})\) be a chain. Then \(K\) has a finite reduction if and only if \((\mathcal{P}, \subseteq)\) has the least element.

Proof. \(\Rightarrow\): Let \(\mathcal{P}_0\) be a finite reduction of \(K = (U, \mathcal{P})\). Then \(\text{ind}\mathcal{P} = \text{ind}\mathcal{P}_0\) and \(\text{ind}\mathcal{P}_0 \neq \text{ind}(\mathcal{P}_0 - \{R\}), \forall R \in \mathcal{P}_0\). Let \(R_1, R_2 \in \mathcal{P}_0\) with \(R_1 \neq R_2\). Since in a chain two elements can be always compared, we can suppose that \(R_1 \subseteq R_2\) without losing generality. Thus

\[
\text{ind}\mathcal{P} = \text{ind}(\mathcal{P} - \{R_2\}) = \text{ind}\mathcal{P}_0 = \text{ind}(\mathcal{P}_0 - \{R_2\}),
\]

contradicting to the assumption that \(\mathcal{P}_0\) is a finite reduction of \(\mathcal{P}\). So, \(\mathcal{P}_0\) has only one element. Let \(\mathcal{P}_0 = \{R_0\}\), then it follows from \(\text{ind}\mathcal{P} = \text{ind}\mathcal{P}_0 = R_0\) that \(R_0\) is the least element in \(\mathcal{P}\).

\(\Leftarrow\): Apply Remark 3.3. \(\Box\)

Definition 3.5. Let \(K = (U, \mathcal{P})\) be a knowledge base. Then \(K^* = (U, \mathcal{P}^*)\) is called the semilattice saturation of \(K\) if \(\mathcal{P}^*\) consists of all the nonempty finite infs of \(\mathcal{P}\).

Notice, even if \((\mathcal{P}, \subseteq)\) is a semilattice, we may have that \(\mathcal{P} \neq \mathcal{P}^*\). If a knowledge base is a semilattice, then itself must be filtered. So, \(K^* = (U, \mathcal{P}^*)\) is filtered and \(\text{ind}\mathcal{P} = \text{ind}\mathcal{P}^*\).

Proposition 3.6. If knowledge base \(K = (U, \mathcal{P})\) is filtered, then \(K\) has a finite reduction iff \(\mathcal{P}\) has the least element.

Proof. \(\Rightarrow\): Let \(\mathcal{P}_0 = \{R_1, \cdots, R_2\}\) be a finite reduction of \(K\). Then \(\text{ind}\mathcal{P}_0 = R_1 \cap \cdots \cap R_n = \text{ind}\mathcal{P}\). Since \(\mathcal{P}\) is filtered, there exists \(R_0 \in \mathcal{P}\) such that \(R_0 \subseteq R_1, \cdots, R_n\). Thus \(R_0 \subseteq \text{ind}\mathcal{P}_0 = \text{ind}\mathcal{P}\). Noticing that \(\text{ind}\mathcal{P} \subseteq R_0\), we have \(R_0 = \text{ind}\mathcal{P}\). So, \(R_0\) is the least element of \(\mathcal{P}\).

\(\Leftarrow\): Apply Remark 3.3. \(\Box\)
Corollary 3.7. If a knowledge base $K = (U, \mathcal{P})$ is a semilattice, then $K$ has a finite reduction iff $\mathcal{P}$ has the least element.

Proof. By Proposition 3.6, we immediately conclude this corollary. \qed

Proposition 3.8. If $K = (U, \mathcal{P})$ is a knowledge base and $K^* = (U, \mathcal{P}^*)$ is the semilattice saturation of $K$, then $K$ has a finite reduction iff $K^*$ has a finite reduction.

Proof. $\Rightarrow$: Let $\mathcal{P}_0$ be a finite reduction of $K$, $K^*(U, \mathcal{P}^*)$ be the semilattice saturation of $K$ and $K^*_0 = (U, \mathcal{P}^*_0)$ the semilattice saturation of $K_0 = (U, \mathcal{P}_0)$. Then $\mathcal{P}_0^* \subseteq \mathcal{P}^*$, and by Corollary 3.7, we conclude that $\mathcal{P}_0^*$ has the least element $R_0 \in \mathcal{P}_0^*$. So, $\text{ind}\mathcal{P} = \text{ind}\mathcal{P}^* \subseteq \text{ind}\mathcal{P}_0^* = \text{ind}\mathcal{P}_0 = \text{ind}\mathcal{P}$. Thus $R_0 = \text{ind}\mathcal{P}_0 = \text{ind}\mathcal{P}_0^* = \text{ind}\mathcal{P}^*$ and $R_0$ is the least element of $\mathcal{P}^*$. By the proof of Remark 3.3, we conclude that $K^*$ has a finite reduction $\{R_0\}$.

$\Leftarrow$: Let $K^* = (U, \mathcal{P}^*)$ be the semilattice saturation of $K$ with a finite reduction. Then $(\mathcal{P}^*, \subseteq)$ is a semilattice. By Corollary 3.7, we conclude that $\mathcal{P}^*$ has the least element $R_0$. So, there are $R_1, \cdots, R_n \in \mathcal{P}$ such that $R_0 = R_1 \cap \cdots \cap R_n$. For the finite knowledge base $(U, \{R_1, \cdots, R_n\})$, by Lemma 2.7, there is a finite reduction $\mathcal{P}_0 \subseteq \{R_1, \cdots, R_n\}$ such that $\text{ind}\mathcal{P}_0 = R_1 \cap \cdots \cap R_n = \text{ind}\mathcal{P}^* = \text{ind}\mathcal{P}$. Since $\mathcal{P}_0 \subseteq \mathcal{P}$ is independent, $\mathcal{P}_0$ is a finite reduction. \qed

By Corollary 3.7 and Proposition 3.8, we immediately have the following

Theorem 3.9. If $K = (U, \mathcal{P})$ is a knowledge base, then $K$ has a finite reduction iff the semilattice saturation $K^* = (U, \mathcal{P}^*)$ of $K$ has the least element.

By Theorem 3.9, we have the following

Corollary 3.10. If $K = (U, \mathcal{P})$ is a knowledge base, then $K$ has a finite reduction iff there are finite elements $R_1, \cdots, R_n \in \mathcal{P}$ such that $R_1 \cap \cdots \cap R_n = \text{ind}\mathcal{P}$.

Proof. $\Rightarrow$: Trivial.

$\Leftarrow$: Since $K_0 = (U, \{R_1, \cdots, R_n\})$ is finite, reduction of $K_0$ always exists. Let $\mathcal{P}_0$ be a reduction of $K_0$. Then $\text{ind}\mathcal{P}_0 = R_1 \cap \cdots \cap R_n$. Noticing that $\mathcal{P}_0$ is independent and $\text{ind}\mathcal{P}_0 = R_1 \cap \cdots \cap R_n = \text{ind}\mathcal{P}$, we have that $\mathcal{P}_0$ is also a finite reduction of $\mathcal{P}$. \qed
4 Several examples

In this section, we are intend to construct examples to show that there is a knowledge base which is a chain but has no reduction, that there is an independent knowledge base which is an infinite anti-chain and that there is a knowledge base which has not only infinite reductions but also finite reductions. The following examples respectively reflect these situations.

Example 4.1. Let $U = \mathbb{N}$ be a universe and $\mathcal{P} = \{R_0, R_1, \cdots, R_i, \cdots\}$ a family of equivalent relations on $U$, where $R_i = \{(0,0),(1,1),\cdots,(i,i)\} \cup \{(k,j)|k,j \geq i+1\}$. Then $\mathcal{P}$ is decreased and has no least element. By Theorem 3.4, $K = (U, \mathcal{P})$ has no finite reduction.

To go further, we assert that $K$ has no reduction. To this end, assume $\mathcal{P}$ has a reduction $\mathcal{P}_0$. Then $\mathcal{P}_0$ must be infinite and $\text{ind}\mathcal{P} = \text{ind}\mathcal{P}_0$. Let $t$ be the least index of $R_i$ such that $R_i \in \mathcal{P}_0$. Then $\mathcal{P}_0 - \{R_i\} \neq \emptyset$. Since $\mathcal{P}$ is decreased, $\text{ind}\mathcal{P}_0 = \text{ind}(\mathcal{P}_0 - \{R_i\})$ and $\mathcal{P}_0$ is not independent, a contradiction to $\mathcal{P}_0$ being a reduction.

Example 4.2. Let $U = \mathbb{N}^+$ be a universe, $\mathcal{P} = \{R_1, \cdots, R_n, \cdots\}$ a family of equivalent relations on $U$, where $R_n = \{(1,n),(n,1),(1,1),\cdots,(n,n)\} \cup \{(k,j)|k,j \geq n+1\}$. Then $R_n$ is an equivalent relation on $U$. Let $K = (U, \mathcal{P})$. We assert that $K$ has no reduction. In fact, $\text{ind}\mathcal{P} = \{(x,x)|x \in U\} = \triangle$, the identity relation on $U$. Any finite meets of $\mathcal{P}$ cannot be $\triangle$. By Theorem 3.10, we see that $\mathcal{P}$ has no finite reduction. Let $R_{i_1}, \cdots, R_{i_k}$ be an infinite sequence of $\mathcal{P}$. For every pair $(i, n)$ with $i \neq n$, pick $i_k > \max\{i, n\}$. Then $(i,n) \notin R_{i_k}$. With this fact, we see that $R_{i_1} \cap \cdots \cap R_{i_k} \cap \cdots = \triangle$. This implies that for any infinite sequence $\mathcal{P}^*$ of $\mathcal{P}$ and any $R \in \mathcal{P}^*$, one has that $\text{ind}\mathcal{P}^* = \text{ind}\mathcal{P} = \text{ind}(\mathcal{P}^* - \{R\}) = \triangle$ and $\mathcal{P}^*$ is not independent. So, $\mathcal{P}$ has no infinite reduction, either.

In this example, $\forall i < n, (1,n) \in R_n, (1,n) \notin R_i, (i + 1, n + 1) \in R_i, (i + 1, n + 1) \notin R_n$. So, $R_n \notin R_i$ and $R_i \notin R_n$. By this fact, we see that $\mathcal{P}$ is an anti-chain.

Example 4.3. Let $U = \mathbb{N}^+$ be a universe, $\mathcal{P} = \{R_1, \cdots, R_n, \cdots\}$ a family of equivalent relations on $U$ such that $U/R_1 = \{\{1\}, \{2,3,\cdots\}\}, U/R_2 = \{\{1,2\}, \{3,4,\cdots\}\}, U/R_3 = \{\{1,2,3\}, \{4,5,\cdots\}\}, \cdots, U/R_n = \{\{1,2,\cdots,n\}, \{n+1, n+2,\cdots\}\}, \cdots$. By Remark 2.3, $R_n$ is indeed an equivalent relation. Let $K = (U, \mathcal{P})$. We will show that the knowledge base $K$ has itself as a reduction. In fact, $\forall i < n$, we have $[1]_{R_i} = \{1,2,\cdots,i\}, [1]_{R_n} = \{1,2,\cdots,n\}$.
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\[ [n + 1]_{R_i} = \{i + 1, i + 2, \cdots\}, [n + 1]_{R_n} = \{n + 1, n + 2, \cdots\}. \] So, \([1]_{R_n} \not\subset [1]_{R_i}\) and \([n + 1]_{R_n} \not\subset [n + 1]_{R_i}\). This reveals that \(\mathcal{P}\) is an anti-chain.

Let \(R_m \in \mathcal{P}\). Then \((m, m+1) \notin R_m\) and when \(i \neq m\), \((m, m+1) \in R_i\). So \((m, m+1) \in \text{ind} (\mathcal{P} - \{R_m\})\). It is easy to see that \(\text{ind} \mathcal{P} = \{(x, x) | x \in U\} = \Delta \neq \text{ind} (\mathcal{P} - \{R_m\})\). So, \(\mathcal{P}\) is independent and \(K\) is a reduction of itself.

**Example 4.4.** If we add another equivalent relation \(R_0 = \{(x, x) | x \in U\} = \Delta\) to \(\mathcal{P}\) in Example 4.3, then we get a new knowledge base \(K^* = (U, \mathcal{P} \cup \{R_0\})\). It is easy to see that \(\mathcal{P}\) and \(\{R_0\}\) are the two reductions of \(K^*\), one is infinite and the other one is finite.

To sum up, in Example 4.1, \(\mathcal{P}\) is a chain with no reduction; in Example 4.2, \(\mathcal{P}\) is an anti-chain with no reduction; in Example 4.3, \(\mathcal{P}\) is an independent anti-chain and has only itself a reduction; and in Example 4.4, \(\mathcal{P}\) has not only a finite reduction but also an infinite reduction.

**References**


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