Common Property (EA) and Common Fixed Point

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Abstract. A Common fixed point theorem was obtained for two pairs of weakly compatible self-maps sharing common property (EA), as a generalization of a result of Singh and Rao.

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INTRODUCTION

Let \((X, d)\) be a metric space.

Definition 1: Self-maps \(f\) and \(S\) on \(X\) are said to satisfy property (EA) \([1]\) if there exist a sequence \(\langle x_n \rangle_{n=1}^{\infty}\) in \(X\) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = z \quad \text{or} \quad \lim_{n \to \infty} d(f x_n, S x_n) = 0
\]

... (1)

Definition 2: Self-maps \(f\) and \(S\) are compatible \([2]\) if \(\lim_{n \to \infty} d(f S x_n, S f x_n) = 0\) whenever \(\langle x_n \rangle_{n=1}^{\infty}\) is a sequence in \(X\) with the choice (1).

If there is no sequence \(\langle x_n \rangle_{n=1}^{\infty}\) such that (1) holds well, the condition of compatibility is vacuously satisfied and \(f\) and \(S\) are said to be vacuously compatible. Therefore nonvacuously
compatible maps and noncompatible maps ensure the existence of a sequence \( \langle x_n \rangle \rightarrow \infty \) in \( X \) with the choice (1). Thus the class of pairs of maps satisfying property (EA) includes the classes of nonvacuously compatible pairs and noncompatible pairs.

The notion of property (EA) was extended to two pairs of self-maps in [4] as follows:

**Definition 3:** Two pairs of self-maps \((f, S)\) and \((g, T)\) are said to share common property (EA) if there exist sequences \( \langle x_n \rangle \rightarrow \infty \) and \( \langle y_n \rangle \rightarrow \infty \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = z \text{ for some } z \in X. \quad \ldots \quad (2)
\]

**Lemma:** Let \( f, g, s \) and \( T \) be self-maps satisfying the condition:

\[
d^2(f x, g y) \leq a \max\{d(S x, f x)d(T y, g y), d(S x, g y)d(T y, f x), d(S x, f x)d(T y, g y), d(S x, g y)d(T y, f x)\}
\]

for all \( x, y \in X \) \quad \ldots \quad (3)

where \( 0 \leq a < 1 \). If the pairs \((f, S)\) and \((g, T)\) satisfy property (EA), then they share the common property (EA).

**Proof.** Suppose that the pair \((f, S)\) satisfies the property (EA). Then there exists \( \langle x_n \rangle \rightarrow \infty \) in \( X \) such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} S x_n = z \) for some \( z \in X \), and the property (EA) of the pair \((g, T)\) guarantees the existence of the sequence \( \langle y_n \rangle \rightarrow \infty \) with \( \lim_{n \to \infty} g y_n = \lim_{n \to \infty} T y_n = z_1 \) for some \( z_1 \in X \).

To share the common property (EA), it requires that \( z = z_1 \).

Taking \( x = x_n, y = y_n \) in the inequality (3) we get

\[
d^2(f x_n, g y_n) \leq a \max\{d(S x_n, f x_n)d(T y_n, g y_n), d(S x_n, g y_n)d(T y_n, f x_n), d(S x_n, g y_n)d(T y_n, g y_n)\}
\]

Now applying the limit as \( n \to \infty \), it follows that \( d^2(z, z_1) \leq a \max\{0, d(z, z_1)d(z, z_1), 0, 0\} \) or \((1-a)d^2(z, z_1)\leq 0\) so that \( z = z_1 \), proving the lemma.

The following is a result of Singh and Rao [6] for two pairs of compatible maps.

**Theorem 1:** Let \( f, g, s \) and \( T \) be self-maps on a complete metric space \( X \) satisfying the inequality (3). Suppose that there exist sequences \( \langle x_n \rangle \rightarrow \infty \) and \( \langle y_n \rangle \rightarrow \infty \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n, S x_n = 0 \quad \text{and} \quad \lim_{n \to \infty} g y_n, T y_n = 0 . \quad \ldots \quad (4)
\]

If the pairs \((f, S)\) and \((g, T)\) are compatible, and both \(S\) and \(T\) are continuous, then all the four self-maps have a unique common fixed point, which is also the unique common fixed point of \(f\) and \(S\) and \(g\) and \(T\).

**Remark 1:** First we acknowledge that condition (4) is the same as the property (EA) of the pairs \((f, S)\) and \((g, T)\), and hence in view of the lemma, it follows that the pairs \((f, S)\) and \((g, T)\) share the common property (EA).

In this paper, we prove that the conclusion of Theorem 1, still holds good even if we restrict the completeness of space \( X \) to either of the subspaces \( S(X) \) and \( T(X) \) in place of the continuity of the maps \( S \) and \( T \), and replacing the compatibility with its weaker version namely weak compatibility (see below) and using the common property (EA).
**Definition 4:** Self-maps which commute at their coincidence points are called weakly compatible [3]. Thus self-maps \( f \) and \( S \) on \( X \) are weakly compatible if \( fSx = Sf x \) whenever \( x \in X \) is such that \( fx = Sx \).

**Remark 2:** Every compatible pair is weakly compatible but the converse is not true [3]. Also it is known that weakly compatibility and the property (EA) are independent.

Our main result is the following

**Theorem 2:** Let \( f, g, S \) and \( T \) be self-maps on \( X \) satisfying the contractive condition (3). Suppose that the pairs \((f, S)\) and \((g, T)\) are weakly compatible and share the common property (EA). Then the self-maps \( f, g, S \) and \( T \) have unique common fixed point, provided either of the following two conditions holds good:

(a) \( S(X) \) is complete and \( f(X) \subseteq T(X) \)

(b) \( T(X) \) is complete and \( g(X) \subseteq S(X) \).

**Proof:** Let \((f, S)\) and \((g, T)\) share common property (EA), given as in (2).

(a) \( S(X) \) is complete and \( f(X) \subseteq T(X) \).

Then from (2) we see that \( Sp = z \) for some \( p \in X \). Writing \( x = p \) and \( y = y_n \) in (3),

\[
d^2(fp, gy_n) \leq \max \{d(Sp, fp) d(Ty_n, gy_n), d(Sp, gy_n) d(Ty_n, fp), d(Sp, fp) d(Ty_n, fp), d(Sp, gy_n) d(Ty_n, gy_n)\}\]

which as \( n \to \infty \) gives

\[
d^2(fp, z) \leq \max \{0, d(Sp, z) d(z, fp), d(z, fp) d(z, fp)\} \leq \max \{0, 0, d(z, fp), d(z, fp)\} \]

or \( fp = z = Sp \). This and the weak compatibility of \( f \) and \( S \) imply that \( fSp = Sfp \) or \( fz = Sz \).

Now using the inclusion \( f(X) \subseteq T(X) \), we can write that \( fz = Tu \) for some \( u \in X \).

Hence \( fz = Sz = Tu \). … (5)

Taking \( x = z \) and \( y = u \) in the inequality (3) and then using (5) we get

\[
d^2(fz, gu) \leq \max \{d(Sz, fz) d(Tu, gu), d(Sz, gu) d(Tu, fz), d(Sz, fz) d(Tu, fz), d(Sz, gu) d(Tu, gu)\}\]

\[
\leq \max \{0, 0, d(fz, gu) d(fz, gu)\} = d^2(fz, gu)
\]

or \( d^2(fz, gu) = 0 \) so that \( fz = gu \). Thus \( fz = Sz = gu = Tu \).

Again, writing \( x = z \) and \( y = y_n \) in (3)

\[
d^2(fz, gy_n) \leq \max \{d(Sz, fz) d(Ty_n, gy_n), d(Sz, gy_n) d(Ty_n, fz), d(Sz, fz) d(Ty_n, fz), d(Sz, gy_n) d(Ty_n, gy_n)\}\].

In the limiting case as \( n \to \infty \), this gives \( d^2(fz, z) \leq ad^2(Sz, z) \) or \( fz = z \) and hence

\( fz = Sz = gu = Tu \). \quad \ldots \quad (6)

Now \( gu = Tu \) and the weak compatibility of \( g \) and \( T \) imply that \( gTu = Tgu \) or \( gz = Tz \). \quad \ldots \quad (7)

With \( x = y = z \), inequality (3) gives

\[
d^2(fz, gz) \leq \max \{d(Sz, fz) d(Tz, gz), d(Sz, gz) d(Tz, fz), d(Sz, fz) d(Tz, fz), d(Sz, gz) d(Tz, gz)\}\]

\[
\leq ad^2(fz, gz)
\]

or \( d^2(fz, gz) = 0 \) so that \( fz = gz \), and finally we get that \( fz = Sz = gz = Tz \), in view of (6) and (7).

Thus \( z \) is a common fixed point of \( f, S, g \) and \( T \).
(b) \( T(X) \) is complete and \( g(X) \subset S(X) \).
Interchanging the roles of \( S \) and \( T \), and of \( f \) and \( g \) in the above proof, one can similarly obtain that \( f, S, g \) and \( T \) have a common fixed point.

**Uniqueness** of the common fixed point follows easily from the inequality (3) by the method of contradiction.

**Remark 2:** The proof of our result (Theorem 2) does not require the continuity of \( S \) and \( T \) unlike that of Theorem 1. In view of Remarks 1 and 2, the common fixed point follows from Theorem 2. It is interesting to note that the completeness of the entire space \( X \) is restricted to only one of the subspaces \( S(X) \) and \( T(X) \).

**References**


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