A Numerical Solution of Classical Van der Pol-Duffing Oscillator by He’s Parameter-Expansion Method

H. Molaei

Department of Mathematics, University Technology Gazi-Tabatabaei, P.O. Box 57169-33959, Urmia, Iran

S. Kheybari

Department of Mathematics, University College of Science and Technology Elm o Fann, Urmia 57351-33746, Iran

Copyright © 2013 H. Molaei and S. Kheybari. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, the classical Van der Pol-Duffing Oscillator is studied. We provide an approximate solution for this equation using parameter expansion method. Also, we obtain approximate values for frequency of the equation. The parameter-expansion method is more efficient than the perturbation method for this equation because the method is independent of perturbation parameter assumption.

Mathematics Subject Classification: 34C15, 39A11

Keywords: Parameter-expansion method; classical Van der Pol-Duffing Oscillator
1 Introduction

The classical Van der Pol-Duffing oscillator appears in many physical problems and is governed by the nonlinear differential equation

\[ \ddot{x} - \mu (1 - x^2) \dot{x} + x + \alpha x^3 = 0; \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \]  

where dots denote time derivative, \( \mu \) and \( \alpha \) are two positive coefficients. It describes electrical circuits and has many applications in science, engineering and also displays a rich variety of nonlinear dynamical behaviors. It generates the limit cycle for small values of \( \mu \), developing into relaxation oscillations when \( \mu \) becomes large which can be evaluated through the Lindstedt’s perturbation method [1].

This paper suggests a universal method to the problem using a new technology called the parameter-expansion method, which is very effective to the problem. We apply the parameter expansion method to obtain approximate solution of (1), also we provide numerical approximation for frequency of \( x \).

2 Parameter expansion method

Parameter-expansion method is an easy and straightforward approach to nonlinear oscillators. Anyone can apply the method to find approximation of amplitude-frequency relationship of an oscillator only with basic knowledge of advance calculus. The basic idea of the parameter expansion method was provided in [2] and one may find several applications of the method over various areas in [3], [4], [5], [6], [7].

To apply parameter-expansion method on (1) we rewrite the equation as

\[ \ddot{x} - \mu \dot{x} + \mu x^2 \dot{x} + 1 + \alpha x^3 = 0, \]  

According to the parameter-expansion method, the solution \( x \) is expanded into a series of an artificial parameter, \( p \) such as

\[ x = x_0 + px_1 + p^2 x_2 + \cdots, \]  

where \( p \) is called a bookkeeping parameter [2]. We also expand all coefficients of the system (2) into a series of \( p \) in a similar way:

\[ \mu = p\mu_1 + p^2 \mu_2 + \cdots, \]

\[ \alpha = p\alpha_1 + p^2 \alpha_2 + \cdots, \]

\[ 1 = \omega^2 + p\omega_1 + p^2 \omega_2 + \cdots. \]
Here $\omega$ is assumed to be the frequency of (2). By substituting the above expansion (3) and (4) into the (2), we have

\[
\begin{align*}
(x_0 + p\dddot{x}_1 + p^2\ddot{x}_2 + \cdots) - (p\mu_1 + p^2\mu_2 + \cdots)(x_0 + p\dot{x}_1 + p^2\dot{x}_2 + \cdots) \\
+ (p\mu_1 + p^2\mu_2 + \cdots)(x_0 + px_1 + p^2x_2 + \cdots)^2(x_0 + p\dot{x}_1 + p^2\dot{x}_2 + \cdots) \\
+ (p\mu_1 + p^2\mu_2 + \cdots)(x_0 + px_1 + p^2x_2 + \cdots) \\
+ (p\mu_1 + p^2\mu_2 + \cdots)(x_0 + px_1 + p^2x_2 + \cdots)^2 = 0.
\end{align*}
\]

Evaluating the terms with the identical powers of $p$, we have

\[
x_0 + \omega^2x_0 = 0,
\]

and

\[
\dddot{x}_1 - \mu_1\ddot{x}_0 + \mu_1x_0\dot{x}_0 + \omega^2x_1 + \omega_1x_0 + \alpha_1x_0^3 = 0.
\]

Solving the (5), we obtain

\[
x_0 = A\sin(\omega t) + B\cos(\omega t),
\]

where $A$ and $B$ are arbitrary constants. Substituting (7) into (6), we obtain

\[
\begin{align*}
\dddot{x}_1 + \omega^2x_1 &= \cos(\omega t)[\mu_1\omega(A - \frac{A^3}{4} - \frac{AB^2}{4}) - \omega_1B - \alpha_1(\frac{3A^2B}{4} + \frac{3B^3}{4})] \\
&+ \sin(\omega t)[\mu_2\omega(-B + \frac{A^2B}{4} + \frac{B^3}{4}) - \omega_1A - \alpha_1(\frac{3A^2B}{4} + \frac{3AB^2}{4})] \\
&+ \cos(3\omega t)[\mu_1\omega(\frac{A^3}{4} - \frac{3AB^2}{4}) - \alpha_1(-\frac{3A^2B}{4} + \frac{B^3}{4})] \\
&+ \sin(3\omega t)[\mu_1\omega(-\frac{3A^2B}{4} + \frac{B^3}{4}) - \alpha_1(-\frac{A^3}{4} + \frac{3AB^2}{4})].
\end{align*}
\]

If the first-order approximation is enough, then, setting $p = 1$ in both equations (3) and (4), we have

\[
x = x_0 + x_1, \quad \mu = \mu_1, \quad \alpha = \alpha_1 \quad \omega_1 = (1 - \omega^2).
\]

Now substituting (9) into (8) yields:

\[
\begin{align*}
\dddot{x}_1 + \omega^2x_1 &= \cos(\omega t)[\mu\omega(A - \frac{A^3}{4} - \frac{AB^2}{4}) - (1 - \omega^2)B - \alpha(\frac{3A^2B}{4} + \frac{3B^3}{4})] \\
&+ \sin(\omega t)[\mu\omega(-B + \frac{A^2B}{4} + \frac{B^3}{4}) - (1 - \omega^2)A - \alpha(\frac{3A^2B}{4} + \frac{3AB^2}{4})] \\
&+ \cos(3\omega t)[\mu\omega(\frac{A^3}{4} - \frac{3AB^2}{4}) - \alpha(-\frac{3A^2B}{4} + \frac{B^3}{4})] \\
&+ \sin(3\omega t)[\mu\omega(-\frac{3A^2B}{4} + \frac{B^3}{4}) - \alpha(-\frac{A^3}{4} + \frac{3AB^2}{4})].
\end{align*}
\]

Eliminating the secular term in $x_1$ requires

\[
\begin{align*}
\mu\omega(A - \frac{A^3}{4} - \frac{AB^2}{4}) - (1 - \omega^2)B - \alpha(\frac{3A^2B}{4} + \frac{3B^3}{4}) &= 0 \\
\mu\omega(-B + \frac{A^2B}{4} + \frac{B^3}{4}) - (1 - \omega^2)A - \alpha(\frac{3A^2B}{4} + \frac{3AB^2}{4}) &= 0,
\end{align*}
\]

and using (11) and (9), we have

\[
\omega = \sqrt{1 + \frac{3}{4}\alpha(A^2 + B^2)},
\]
and the frequency of equation is

\[ T = \frac{2\pi}{\omega} = \sqrt{\frac{2\pi}{1 + \frac{3}{4}\alpha(A^2 + B^2)}}. \]

Furthermore, (10) can be simplified as

\[ \ddot{x}_1 + \omega^2 x_1 = \cos(3\omega t)[\mu\omega\left(\frac{A^3}{4} - \frac{3AB^2}{4}\right) - \alpha\left(-\frac{3A^2B}{4} + \frac{B^3}{4}\right)] \\
+ \sin(3\omega t)[\mu\omega\left(-\frac{3A^2B}{4} + \frac{B^3}{4}\right) - \alpha\left(-\frac{A^3}{4} + \frac{3AB^2}{4}\right)]. \tag{12} \]

Solving (12) yields

\[ x_1 = \frac{1}{32}\left(\frac{\alpha}{\omega^2}(3AB^2 - A^3) - \frac{\mu}{\omega}(B^3 - 3A^2B)\right) \sin(3\omega t) \\
+ \frac{1}{32}\left(\frac{\alpha}{\omega^2}(B^3 - 3A^2B) - \frac{\mu}{\omega}(A^3 - 3AB^2)\right) \cos(3\omega t). \tag{13} \]

Now using (7), (9) and (13) we obtain the first order solution for \( x \)

\[ x = x_0 + x_1 = A\sin\left(\sqrt{1 + \frac{3}{4}\alpha(A^2 + B^2)}t\right) + B\cos\left(\sqrt{1 + \frac{3}{4}\alpha(A^2 + B^2)}t\right) \\
+ \frac{1}{32}\left(\frac{\alpha}{1 + \frac{3}{4}\alpha(A^2 + B^2)}(3AB^2 - A^3) - \frac{\mu}{1 + \frac{3}{4}\alpha(A^2 + B^2)}(B^3 - 3A^2B)\right) \sin(3\sqrt{1 + \frac{3}{4}\alpha(A^2 + B^2)}t) \\
+ \frac{1}{32}\left(\frac{\alpha}{1 + \frac{3}{4}\alpha(A^2 + B^2)}(B^3 - 3A^2B) - \frac{\mu}{1 + \frac{3}{4}\alpha(A^2 + B^2)}(A^3 - 3AB^2)\right) \cos(3\sqrt{1 + \frac{3}{4}\alpha(A^2 + B^2)}t). \]

Figure (1) show the plot of approximate value for \( x \).
3 Conclusion

In this paper, we studied the classical Van der Pol-Duffing Oscillator. We obtained approximate value for frequency of the equation. On account of, He’s parameter-expansion method is an efficient method to solve such equations. The procedure of looking for a solution is also very simple and straightforward.

ACKNOWLEDGEMENTS. In this study, we applied the parameter-expansion method to solve the classical Van der Pol-Duffing Oscillator. The method more efficient than perturbation methods for this problem because the method is independent of perturbation parameter assumption and one iteration step provide an approximate solution which is valid for the whole solution domain.

References


Received: May 1, 2013