Open Neighborhood Coloring of Graphs

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Abstract

For a simple, connected, undirected graph $G(V, E)$ an open neighborhood coloring of the graph $G$ is a mapping $f : V(G) \rightarrow \mathbb{Z}^+$ such that for each $w \in V$, and $\forall u, v \in N(w)$, $f(u) \neq f(v)$. The maximum value of $f(w), \forall w \in V(G)$ is called the span of the open neighborhood coloring $f$. The minimum value of span of $f$ over all open neighborhood colorings $f$ is called open neighborhood chromatic number of $G$, denoted by $\chi_{onc}(G)$. In this paper we determine the open neighborhood chromatic number of some standard graphs, trees and the infinite triangular lattice.
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1 Introduction

All the graphs considered here are undirected, connected and simple. We use standard terminologies and the terms not defined here may be found in [2, 7, 6]. A k-coloring of a graph \( G \) is a coloring \( f : V(G) \to S \), where \( S \subset \mathbb{Z}^+ \) and \( |S| = k \). The elements of \( S \) are called colors. A \( k \)-coloring is proper if every pair of adjacent vertices receives different colors. A graph is \( k \)-colorable if it has a proper \( k \)-coloring. The chromatic number \( \chi(G) \) is the least \( k \) such that \( G \) is \( k \)-colorable. For results on proper coloring, we refer [6].

We define open neighborhood coloring of a graph \( G(V, E) \) as a coloring \( f : V(G) \to \mathbb{Z}^+ \), such that for each \( w \in V(G) \) and \( \forall u, v \in N(w), f(u) \neq f(v) \). The maximum value of \( f(w), \forall w \in V(G) \) is called the span of the open neighborhood coloring \( f \). The minimum value of span of \( f \) over all open neighborhood colorings \( f \) is called open neighborhood chromatic number of \( G \), denoted by \( \chi_{onc}(G) \).

![Figure 1: A proper coloring and an open neighborhood coloring of the graph \( G \)](image)

Remark 1.1. An open neighborhood coloring of a graph \( G \) need not be a proper coloring of \( G \).

The definition of open neighborhood coloring of a graph \( G \) is motivated by the definition of \( L(h, k) \) labeling [14]. Given a graph \( G(V, E) \), and two non-negative integers \( h \) and \( k \), an \( L(h, k) \) labeling is an assignment of non
negative integers to the vertices of $G$ such that adjacent vertices are labeled using colors at least $h$ apart, and vertices having a common neighbor are labeled using colors at least $k$ apart. This definition imposes a condition on labels of vertices connected by a 2-length path instead of using concept of distance 2, which is very common in the literature. When $h \geq k$, the two definitions coincide, but are different when $h < k$. For example when $h < k$, the former definition of $L(h, k)$ labeling requires that the vertices of a triangle be differently labeled where as the latter definition does not as shown in the Figure 2 and 3. Most of the work in literature deals with the latter definition of $L(h, k)$ labeling with $h \geq k$ [3, 4, 8, 11, 10, 5, 13]. The $L(h, k)$ labeling problem is studied mainly to avoid hidden terminal interference in multihop radio networks in [1] and for code assignment in computer networks in [9]. Our definition of open neighborhood coloring is found to be equivalent to the former definition of $L(h, k)$ labeling dealing with the case of $h < k$, choosing $h = 0$ and $k = 1$, which has been rarely dealt with. The $L(0, 1)$ labeling of a special class of graphs called cactus graphs has been studied in [12]. In this paper, we determine the open neighborhood chromatic number of some standard graphs and the triangular lattice which is an infinite graph. By the definition of open neighborhood coloring, the coloring starts from 1 instead of 0.

2 Some preliminary results

**Observation 2.1.** By the definition it follows that if $f$ is an open neighborhood coloring of $G(V, E)$ with $\text{span}(f) = \chi_{\text{onc}}(G)$, then $f(u) \neq f(v)$ holds if and only if the vertices $u, v$ are the end vertices of a path of length 2 in $G$.

**Observation 2.2.** For any graph $G(V, E)$, $\chi_{\text{onc}}(G) \geq \Delta(G)$.

**Observation 2.3.** If $H$ is a connected subgraph of $G$, then $\chi_{\text{onc}}(H) \leq \chi_{\text{onc}}(G)$.

**Observation 2.4.** The open neighborhood chromatic number of a connected graph $G$, is 1 if and only if $G \cong K_1$ or $K_2$. 

Figure 2: An $L(0,1)$ labeling where vertices at distance 2 are labeled differently.

Figure 3: An $L(0,1)$ labeling where vertices connected by a path of length 2 are labeled differently.
Theorem 2.5. If $G$ is a triangle free graph with $n$ vertices then $\chi_{\text{onc}}(G) \leq n - 1$.

Proof. We prove the theorem in two cases.

Case 1: Suppose there exists a vertex $v$ of degree $n - 1$. Then $G \cong K_{1,n-1}$ as $G$ is triangle free. In this case $\chi_{\text{onc}}(G) \geq n - 1$. Further assigning one color to each of the $n - 1$ vertices adjacent to $v$ and repeating one of these colors to $v$, gives an open neighborhood coloring of $G$. Thus $\chi_{\text{onc}}(G) = n - 1$.

Case 2: Suppose there exists no vertex $v$ of degree $n - 1$. Then every vertex is adjacent to at most $n - 2$ vertices of $G$. In this case assigning one color each to the $n - 2$ vertices adjacent to any vertex and assigning the $(n - 1)^{th}$ color to the vertex gives an open neighborhood coloring of $G$. Thus $\chi_{\text{onc}}(G) \leq n - 1$.

Theorem 2.6. Let $G$ be any graph and $v_0$ be a vertex of maximum degree $\Delta(G)$. Suppose $H$ is a subgraph induced by $N[v_0]$ such that every $v \in N(v_0)$ belongs to a triangle containing $v_0$. Then, $\chi_{\text{onc}}(G) \geq \Delta(G) + 1$.

Proof. By Observation 2.2, $\chi_{\text{onc}}(G) \geq \Delta(G)$. Further $v_0$ cannot be assigned any of the colors assigned to its neighbors $v$, as $v_0$ and $v$ belong to some triangle in $H$ and hence have a common neighbor. Therefore $\chi_{\text{onc}}(H) \geq \deg(v_0) + 1 = \Delta(G) + 1$. By Observation 2.3, $\chi_{\text{onc}}(G) \geq \chi_{\text{onc}}(H)$. Hence $\chi_{\text{onc}}(G) \geq \chi_{\text{onc}}(H) \geq \Delta(G) + 1$.

Theorem 2.7. Let $G(V,E)$ be any graph and $G_1(V,E')$ be the graph obtained by adding an edge between every pair of vertices $u$ and $v$ of $G$ that are connected by a path of length two. Then, $\chi_{\text{onc}}(G) \geq \chi(G_1)$.

Proof. In an open neighborhood coloring of a graph $G$ we assign different colors to the vertices $u$ and $v$ which are connected by a path of length two in $G$. Since these two are adjacent in $G_1$ an open neighborhood coloring of $G$ is nothing but a proper coloring of $G_1$ and $\chi(G_1)$ is the minimum number of colors required for a proper coloring of $G_1$. Hence the result.

Theorem 2.8. If $G$ is a graph of diameter 2 whose complement $\bar{G}$ is connected then, $\chi_{\text{onc}}(G) \geq \chi(\bar{G})$.

Proof. Let $u, v \in V(G)$ such that $d_G(u, v) = 2$. Then there exists a path of length 2 between them in $G$. So any open neighborhood coloring of $G$ assigns different colors to $u$ and $v$. Now $d_G(u, v) = 1$ and hence an open neighborhood coloring of $G$ is a proper coloring of $\bar{G}$. But $\chi(\bar{G})$ is the minimum number of colors required for a proper coloring of $G$. Hence $\chi_{\text{onc}}(G) \geq \chi(\bar{G})$. 

$\square$
3 Open Neighborhood Coloring of some standard Graphs

**Theorem 3.1.** If \( G \cong K_n, n \geq 3 \) or \( K_1 + P_{n-1}, n \geq 3 \) or \( W_{1,n-1}, n \geq 4 \) then \( \chi_{onc}(G) = |V(G)| \).

**Proof.** Every pair of vertices in \( G \), are the end vertices of some path of length 2 in \( G \). By the Observation 2.1, every pair of vertices in \( G \) must be given a different color. Hence the result. \(\square\)

**Theorem 3.2.** For a path \( P_n, n \geq 2 \),

\[
\chi_{onc}(P_n) = \begin{cases} 
1, & \text{if } n = 2, \\
2, & \text{if } n \geq 3.
\end{cases}
\]

**Proof.** Let \( P_n \) be a path on \( n \geq 2 \) vertices with \( V(P_n) = \{v_1, v_2 \ldots v_n\} \). The case of \( n = 2 \), has been dealt in Observation 2.4.

We now consider the case of \( n \geq 3 \). As \( \Delta(P_n) = 2 \), by Observation 2.2 we have \( \chi_{onc}(P_n) \geq 2 \). We now show that \( \chi_{onc}(P_n) = 2 \), by defining the coloring \( f : V(P_n) \to Z^+ \) as

\[
f(v_i) = \begin{cases} 
1, & \text{if } i \equiv 1,2 \pmod{4} \\
2, & \text{if } i \equiv 0,3 \pmod{4}.
\end{cases}
\]

Now for all \( i, 2 \leq i \leq n - 1 \), \( N(v_i) = \{v_{i-1}, v_{i+1}\} \).

If \( i \equiv 0 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).

If \( i \equiv 1 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).

If \( i \equiv 2 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

If \( i \equiv 3 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

Therefore \( f(v_{i-1}) \neq f(v_{i+1}), \forall i, 2 \leq i \leq n - 1 \).

Hence \( \chi_{onc}(P_n) = 2 \) when \( n \geq 3 \). \(\square\)

**Theorem 3.3.** For a cycle \( C_n, n \geq 3 \),

\[
\chi_{onc}(C_n) = \begin{cases} 
2, & \text{if } n \equiv 0 \pmod{4}, \\
3, & \text{otherwise}.
\end{cases}
\]

**Proof.** Let \( C_n \) be a cycle on \( n \geq 3 \) vertices with \( V(C_n) = \{v_0, v_1 \ldots v_{n-1}\} \). The case of \( n = 3 \), has been dealt in Theorem 3.1.

We now consider the case of \( n \geq 4 \) in four cases.

As \( \Delta(C_n) = 2 \), by Observation 2.2 we have \( \chi_{onc}(C_n) \geq 2 \).

**Case(1):** \( n \equiv 0 \pmod{4} \)

For all \( i, 0 \leq i \leq n - 1 \), we define the coloring \( f : V(C_n) \to Z^+ \) as
\[ f(v_i) = \begin{cases} 
1, & \text{if } i \equiv 0,1 \pmod{4} \\
2, & \text{if } i \equiv 2,3 \pmod{4}. 
\end{cases} \]

Now for all \( i, 0 \leq i \leq n-1 \), \( N(v_i) = \{v_{i-1}, v_{i+1}\} \), where the suffix is under modulo \( n \).

If \( i \equiv 0 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).

If \( i \equiv 1 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

If \( i \equiv 2 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

If \( i \equiv 3 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).

Therefore for all \( i, 0 \leq i \leq n-1 \), \( f(v_{i-1}) \neq f(v_{i+1}) \).

Hence \( \chi_{onc}(C_n) = 2 \) when \( n \geq 4, n \equiv 0 \pmod{4} \).

Let \( C_n'(V', E') \) be the graph with \( V' = V(C_n) \) and \( E' = \{v_iv_j : v_i \text{ and } v_j \text{ are end vertices of a path of length } 2 \text{ in } C_n\} \). Then by Theorem 2.7, \( \chi_{onc}(C_n) \geq \chi(C_n') \).

We now consider the cases of \( n \equiv 1, 2, 3 \pmod{4} \) and show in each case that \( \chi(C_n') = 3 \) and hence \( \chi_{onc}(C_n) \geq 3 \).

**Case (2):** \( n \equiv 1 \pmod{4} \)

For \( C_n, n \equiv 1 \pmod{4} \), \( C_n'(V', E') \) is an odd cycle with \( V' = V(C_n) \) and 
\( E' = \{v_0v_2, v_2v_4, v_4v_6 \ldots v_{n-1}v_1, v_1v_3, \ldots v_{n-2}v_0\} \) and any odd cycle requires three colors for a proper coloring. Therefore \( \chi(C_n') = 3 \) and hence \( \chi_{onc}(C_n) \geq 3 \).

We show that \( \chi_{onc}(C_n) = 3 \).

For all \( i, 0 \leq i \leq n-2 \), we define the coloring \( f : V(C_n) \to \mathbb{Z}^+ \) as
\[ f(v_i) = \begin{cases} 
1, & \text{if } i \equiv 0,1 \pmod{4} \\
2, & \text{if } i \equiv 2,3 \pmod{4} 
\end{cases} \]

and
\[ f(v_{n-1}) = 3 \]

Now for all \( i, 0 \leq i \leq n-1 \), \( N(v_i) = \{v_{i-1}, v_{i+1}\} \), where the suffix is under modulo \( n \).

For all \( i, 1 \leq i \leq n-3 \),

If \( i \equiv 0 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).

If \( i \equiv 1 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

If \( i \equiv 2 \pmod{4} \) then \( f(v_{i-1}) = 1 \neq 2 = f(v_{i+1}) \).

If \( i \equiv 3 \pmod{4} \) then \( f(v_{i-1}) = 2 \neq 1 = f(v_{i+1}) \).
Also, 
\[ N(v_0) = \{v_1, v_{n-1}\} \text{ and } f(v_1) = 1 \neq 3 = f(v_{n-1}). \]
\[ N(v_{n-2}) = \{v_{n-3}, v_{n-1}\} \text{ and } n \equiv 1 \pmod{4} \Rightarrow f(v_{n-3}) = 2 \neq 1 = f(v_{n-1}). \]
\[ N(v_{n-1}) = \{v_{n-2}, v_0\} \text{ and } n \equiv 1 \pmod{4} \Rightarrow f(v_{n-2}) = 2 \neq 1 = f(v_0). \]
Therefore for all \(i, 0 \leq i \leq n-1\), \(f(v_i) \neq f(v_{i+1})\).
Hence \(\chi_{onc}(C_n) = 3\) when \(n \geq 4\) and \(n \equiv 1 \pmod{4}\).

**Case(3):** \(n \equiv 2 \pmod{4}\)
In this case \(C''_n(V', E')\) is a disjoint union of two odd cycles, \(C''_{n_1}(V'_1, E'_1)\) and \(C''_{n_2}(V'_2, E'_2)\) where \(V'_1 = \{v_0, v_2 \ldots v_{n-2}\}, E'_1 = \{v_0v_2, v_2v_4 \ldots v_{n-2}v_0\}\) and \(V'_2 = \{v_1, v_3 \ldots v_{n-1}\}, E'_2 = \{v_1v_3, v_3v_5 \ldots v_{n-1}v_1\}\). As \(C''_{n_1}(V'_1, E'_1)\) is an odd cycle \(\chi(C''_{n_1}) = 3\). The same colors can be used to properly color the odd cycle \(C''_{n_2}(V'_2, E'_2)\) as the two cycles are disjoint. Hence \(\chi(C''_n) = 3\) and by Theorem 2.7 \(\chi_{onc}(C_n) \geq \chi(C''_n) = 3\). We now show that \(\chi_{onc}(C_n) = 3\).
For all \(i, 0 \leq i \leq n-3\), we define the coloring \(f : V(C_n) \rightarrow Z^+\) as
\[ f(v_i) = \begin{cases} 1, & \text{if } i \equiv 0, 1 \pmod{4} \\ 2, & \text{if } i \equiv 2, 3 \pmod{4}. \end{cases} \]
and
\[ f(v_{n-2}) = f(v_{n-1}) = 3. \]
Now for all \(i, 0 \leq i \leq n-1\), \(N(v_i) = \{v_{i-1}, v_{i+1}\}\), where the suffix is under modulo \(n\).
For all \(i, 1 \leq i \leq n-4\),
If \(i \equiv 0 \pmod{4}\) then \(f(v_{i+1}) = 2 \neq 1 = f(v_{i+1})\).
If \(i \equiv 1 \pmod{4}\) then \(f(v_{i-1}) = 1 \neq 2 = f(v_{i+1})\).
If \(i \equiv 2 \pmod{4}\) then \(f(v_{i-1}) = 1 \neq 2 = f(v_{i+1})\).
If \(i \equiv 3 \pmod{4}\) then \(f(v_{i-1}) = 2 \neq 1 = f(v_{i+1})\).
Also,
\[ N(v_0) = \{v_1, v_{n-1}\} \text{ and } f(v_1) = 1 \neq 3 = f(v_{n-1}). \]
\[ N(v_{n-3}) = \{v_{n-4}, v_{n-2}\} \text{ and } n \equiv 2 \pmod{4} \Rightarrow f(v_{n-4}) = 2 \neq 1 = f(v_{n-2}). \]
\[ N(v_{n-2}) = \{v_{n-3}, v_{n-1}\} \text{ and } n \equiv 2 \pmod{4} \Rightarrow f(v_{n-3}) = 2 \neq 1 = f(v_{n-1}). \]
\[ N(v_{n-1}) = \{v_{n-2}, v_0\} \text{ and } n \equiv 1 \pmod{4} \Rightarrow f(v_{n-2}) = 2 \neq 1 = f(v_0). \]
Therefore for all \(i, 0 \leq i \leq n-1\), \(f(v_{i+1}) \neq f(v_{i+1})\).
Hence \(\chi_{onc}(C_n) = 3\) when \(n \geq 4\) and \(n \equiv 2 \pmod{4}\).

**Case(4):** \(n \equiv 3 \pmod{4}\)
For \(C_n, n \equiv 3 \pmod{4}\), \(C''_n(V', E')\) is an odd cycle with \(V' = V(C_n)\) and \(E' = \{v_0v_2, v_2v_4, v_4v_6 \ldots v_{n-1}v_{n-1}, v_1v_3, \ldots v_{n-2}v_0\}\) and any odd cycle requires three colors for a proper coloring. Therefore \(\chi(C''_n) = 3\) and hence by Theorem 2.7 \(\chi_{onc}(C_n) \geq 3\). We now show that \(\chi_{onc}(C_n) = 3\).
For all \(i, 1 \leq i \leq n-1\), we define the coloring
Also, for all $j$,

$$f(v_i) = \begin{cases} 
1, & \text{if } i \equiv 0,1 \pmod{4} \\
2, & \text{if } i \equiv 2,3 \pmod{4}.
\end{cases}$$

and

$$f(v_0) = 3$$

Now for all $i, 1 \leq i \leq n - 1$, $N(v_i) = \{v_{i-1}, v_{i+1}\}$, where the suffix is under modulo $n$.

For all $i, 2 \leq i \leq n - 2$

If $i \equiv 0 \pmod{4}$ then $f(v_{i-1}) = 2 \not= 1 = f(v_{i+1})$.

If $i \equiv 1 \pmod{4}$ then $f(v_{i-1}) = 1 \not= 2 = f(v_{i+1})$.

If $i \equiv 2 \pmod{4}$ then $f(v_{i-1}) = 1 \not= 2 = f(v_{i+1})$.

If $i \equiv 3 \pmod{4}$ then $f(v_{i-1}) = 2 \not= 1 = f(v_{i+1})$.

Also, $N(v_0) = \{v_{n-1}, v_1\}$ and $n \equiv 3 \pmod{4} \Rightarrow f(v_{n-1}) = 2 \not= 1 = f(v_1)$.

$N(v_1) = \{v_0, v_2\}$ and $f(v_0) = 1 \not= 2 = f(v_2)$.

Further, $N(v_{n-1}) = \{v_{n-2}, v_0\}$ and $n \equiv 3 \pmod{4} \Rightarrow f(v_{n-2}) = 1 \not= 3 = f(v_0)$.

Therefore for all $i, 0 \leq i \leq n - 1$, $f(v_{i-1}) \not= f(v_{i+1})$.

Hence $\chi_{\text{onc}}(C_n) = 3$ when $n \geq 4$ and $n \equiv 3 \pmod{4}$.

Hence the result. \hfill \Box

**Theorem 3.4.** For a complete bipartite graph $K_{m,n}$ where $m, n \geq 2$, $\chi_{\text{onc}}(K_{m,n}) = \max\{m, n\}$.

**Proof.** Let $V(K_{m,n}) = V_1 \cup V_2$, with $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$, where $m, n \geq 2$ Without loss of generality, we may assume $m = \max\{m, n\}$.

(The proof for the case of $n = \max\{m, n\}$ is similar). Therefore, $\Delta(K_{m,n}) = m$.

By Observation 2.2, we have, $\chi_{\text{onc}}(K_{m,n}) \geq m$.

We now prove that $\chi_{\text{onc}}(K_{m,n}) = m$ by defining the coloring $f : V(K_{m,n}) \to Z^+$ as follows:

For all $i, 1 \leq i \leq m$, $f(v_i) = i$ and for all $j, 1 \leq j \leq n$, $f(u_j) = j$.

Now, for all $i, 1 \leq i \leq m$, $N(v_i) = \{u_1, u_2, \ldots, u_n\}$ and $f(u_l) = l \not= k = f(u_k)$ for all $1 \leq l, k \leq n$.

Also, for all $j, 1 \leq j \leq n$, $N(u_i) = \{v_1, v_2, \ldots, v_m\}$ and $f(v_l) = l \not= k = f(v_k)$ for all $1 \leq l, k \leq m$. Therefore, $f$ is an open neighborhood coloring which uses a minimum of $m$ colors as $m = \max\{m, n\}$. Hence the result. \hfill \Box

**Theorem 3.5.** For a tree $T$ on $n \geq 3$ vertices, $\chi_{\text{onc}}(T) = \Delta(T)$.

**Proof.** By Observation 2.2, we have $\chi_{\text{onc}}(T) \geq \Delta(T)$. We now show that $\Delta(T)$ colors are sufficient for an open neighborhood coloring of $T$. Let $v_0$ be
the vertex of maximum degree in $T$. Let $\deg(v_0) = \Delta(T) = k$. Consider the tree to be rooted at $v_0$. Let $N(v_0) = \{v_1, v_2, \ldots, v_k\}$. These $k$ vertices must be assigned $k$ different colors say $c_1, c_2, \ldots, c_k$. Any one of these colors say $c_1$ may be assigned to $v_0$. Now consider an arbitrary vertex say $v_i \in N(v_0)$. Now $\deg(v_i) \leq k$. As $T$ is a tree, $N(v_0) \cap N(v_i) = \emptyset$. Let $S = N(v_i) - v_0$. Thus $|S| \leq k - 1$. Any $v_j \in S$ is connected to $v_0$ by a path of length 2 and hence cannot be assigned the color $c_1$. Hence we are left with $k - 1$ colors which can be reused to color the vertices of the set $S$. Hence $k = \Delta(T)$ colors are sufficient for an open neighborhood coloring of $T$. \hfill $\Box$

**Theorem 3.6.** Let $G(V, E)$ be a connected graph on $n \geq 3$ vertices. Then $\chi_{\text{onec}}(G) = n$ if and only if $N(u) \cap N(v) \neq \emptyset$ holds for every pair of vertices $u, v \in V(G)$.

**Proof.** Let $u, v \in V(G)$ be any two arbitrary vertices of $G$ such that $N(u) \cap N(v) \neq \emptyset$.

$\Rightarrow$ There exists $w \in V(G)$, such that $w \in N(u) \cap N(v)$.

$\Rightarrow w \in N(u)$ and $w \in N(v)$

$\Rightarrow u \in N(w)$ and $v \in N(w)$

$\Rightarrow f(u) \neq f(v)$.

Hence every pair of vertices in $G$ are assigned different colors. Therefore $\chi_{\text{onec}}(G) = n$.

We now prove the converse part. Suppose $N(u) \cap N(v) = \emptyset$ holds for some pair of vertices $u, v \in V(G)$.

$\Rightarrow$ there exists no vertex $w \in V(G)$ such that $w \in N(u) \cap N(v)$

$\Rightarrow$ there exists no vertex $w \in V(G)$ such that $w \in N(u)$ and $w \in N(v)$.

$\Rightarrow$ there exists no vertex $w \in V(G)$ such that $u, v \in N(w)$.

$\Rightarrow$ there exists an open neighborhood coloring $f : V(G) \rightarrow \mathbb{Z}^+$ with $f(u) = f(v)$.

$\Rightarrow \chi_{\text{onec}}(G) \leq n - 1$ a contradiction. Hence the result. \hfill $\Box$

## 4 Open neighborhood coloring of a triangular lattice

In this section we determine the open neighborhood chromatic number of the infinite triangular lattice which is defined as follows : Let $\varepsilon_1 = (1, 0), \varepsilon_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ be two vectors in Euclidean plane. The triangular lattice is the infinite graph $\Gamma_\Delta$ with vertex set $V(\Gamma_\Delta) = \{i\varepsilon_1 + j\varepsilon_2 : i, j \in \mathbb{Z}\}$ and $E(\Gamma_\Delta) = \{uv : u, v \in V(\Gamma_\Delta), d_E(u, v) = 1\}$, where $d_E(u, v)$ is the euclidean distance between $u$ and $v$. In short we denote the vertices by $(i, j)$. 
The triangular lattice plays an important role in radio broadcasting and mobile cellular networks. In a mobile radio network, large service areas are often covered by a network of congruent polygonal cells, with each transmitter in the center of the cell that it covers. A honeycomb of hexagonal cells provides the most economic covering and here the transmitters are placed in a triangular lattice.

![Figure 4: Triangular lattice $\Lambda_\Delta$](image)

Figure 4: Triangular lattice $\Lambda_\Delta$

![Figure 5: $v \cup N_1(v) \cup N_2(v)$](image)

Figure 5: $v \cup N_1(v) \cup N_2(v)$

**Theorem 4.1.** The open neighborhood coloring of the triangular lattice is 7.

**Proof.** For any vertex $v \in V(\Gamma_\Delta)$, let $N_1(v)$ be the neighbors of $v_1$ and $N_2(v)$ be the set of vertices that are connected to $v$ by a path of length 2. Then, for $v = (i, j) \in V(\Gamma_\Delta)$, $N_1((i, j)) = \{(i \pm 1, j), (i, j \pm 1), (i \pm 1, j \mp 1)\}$ and $N_2((i, j)) = \{(i \pm 2, j), (i, j \pm 2), (i \pm 1, j \mp 2), (i \pm 2, j \mp 2), (i \pm 1, j \pm 1), (i \pm 2, j \mp 1)\}$. The Figure 5 shows the induced subgraph of $v \cup N_1(v) \cup N_2(v)$.

The induced subgraph of $v \cup N_1(v)$, is a wheel $W_{1,6}$. Then by Theorem 3.1, $\chi_{onc}(W_{1,6}) = 7$. Also by Observation 2.3 $\chi_{onc}(\Gamma_\Delta) \geq \chi_{onc}(W_{1,6}) = 7$. We now show that $\chi_{onc}(\Gamma_\Delta) = 7$. 
Define the coloring \( f : V(\Gamma) \to \mathbb{Z}^+ \) as:

\[
f(i, j) = \begin{cases} 
(i + 3) \pmod{7}, & \text{if } i \equiv 0, 1, 2, 3, 5, 6 \pmod{7}, j \equiv 0 \pmod{7} \\
7, & \text{if } i \equiv 4 \pmod{7}, j \equiv 0 \pmod{7} \\
(i + 1) \pmod{7}, & \text{if } i \equiv 0, 1, 2, 3, 4, 5 \pmod{7}, j \equiv 1 \pmod{7} \\
7, & \text{if } i \equiv 6 \pmod{7}, j \equiv 1 \pmod{7} \\
(i - 1) \pmod{7}, & \text{if } i \equiv 0, 2, 3, 4, 5, 6 \pmod{7}, j \equiv 2 \pmod{7} \\
7, & \text{if } i \equiv 1 \pmod{7}, j \equiv 2 \pmod{7} \\
(i - 3) \pmod{7}, & \text{if } i \equiv 0, 1, 2, 4, 5, 6 \pmod{7}, j \equiv 3 \pmod{7} \\
7, & \text{if } i \equiv 3 \pmod{7}, j \equiv 3 \pmod{7} \\
(i + 2) \pmod{7}, & \text{if } i \equiv 0, 1, 2, 3, 4, 6 \pmod{7}, j \equiv 4 \pmod{7} \\
7, & \text{if } i \equiv 5 \pmod{7}, j \equiv 4 \pmod{7} \\
i \pmod{7}, & \text{if } i \equiv 1, 2, 3, 4, 5, 6 \pmod{7}, j \equiv 5 \pmod{7} \\
7, & \text{if } i \equiv 0 \pmod{7}, j \equiv 5 \pmod{7} \\
(i - 2) \pmod{7}, & \text{if } i \equiv 0, 1, 3, 4, 5, 6 \pmod{7}, j \equiv 6 \pmod{7} \\
7, & \text{if } i \equiv 2 \pmod{7}, j \equiv 6 \pmod{7}
\end{cases}
\]

The vertices \( N_1(v) \) and \( N_2(v) \) are the only vertices connected to \( v \) by a path of length 2. By the coloring defined above none of these vertices receive the color of \( v \), as shown in the Figures 6 and 7 when \( f(v) = 1 \) and \( f(v) = 2 \) respectively. A similar coloring can be obtained when \( f(v) \) is assigned any color from 3 through 7 by cyclically changing the colors 1 to 2, 2 to 3 and so on up to 7 to 1. In all these colorings the largest color used is 7. Hence \( \chi_{\text{once}}(\Gamma) = 7 \).
References


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