On Ginsberg Theorem in Base $b$

Abdulkareem Hamarsheh

Department of Mathematics
Al-Hussein Bin Talal University, Ma’an-Jordan
abkmsaleh@yahoo.com

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Abstract

In 2004, B. Ginsberg proved that if the period of a reciprocal of a prime $p \geq 5$ has length $r = 3w$ and is split into three pieces then their sum is a string of 9’s. In this note we give a very simple proof of the generalized theorem to any base $b$.

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1 INTRODUCTION

According to Dickson [6], E. Midy proved in 1836 that if the period of a reciprocal of a prime $p \geq 5$ has even length and is split into two half-periods then the sum of the halves is a string of 9’s. For example, $\frac{1}{7} = 0.142857$ with $142 + 857 = 999$, and $\frac{1}{17} = 0.0588235294117647$ with $05882352 + 94117647 = 99999999$. Midy’s theorem holds also for reciprocals of some composite numbers. For example, $\frac{1}{11} = 0.090909$ with $090 + 909 = 999$, and $\frac{1}{121} = 0.0082644628099173553719$ with $00826446280 + 99173553719 = 99999999999$. 

In 2004, B. Ginsberg [3], generalized Midy’s theorem to decimal expansions with period \( r = 3w \). For example, \( \frac{1}{7} = 0.142857 \) with \( 14+28+57=99 \), and \( \frac{1}{37} = 0.032258064516129 \) with

\[
03225+80645+16129=99999.
\]

Ginsberg theorem holds also for reciprocals of some composite numbers. For example, \( \frac{1}{57} = 0.017543859649122807 \) with

\[
017543+859649+122807=999999.
\]

## 2 GINSBERG THEOREM

Many authors have given proofs of Midy’s theorem; see, for example, [2],[1],[7],[5], and [4]. In this paper we will give a very simple proof of Ginsberg theorem, generalized to any base \( b \), based only on modular arithmetic.

We begin with the following lemma:

**Lemma 1.** Let \( m \) be relatively prime to the base \( b \). Suppose that the period in base \( b \) of the reciprocal of \( m \) has length \( r = 3w \): \( \frac{1}{m} = 0.[a_1a_2...a_{3w}]_b \).

If \( 1 + b^w + b^{2w} \equiv 0 \mod m \), then \( a_w + a_{2w} + a_{3w} \equiv b - 1 \mod b \)

and \( a_i + a_{w+i} + a_{2w+i} = b - 2b - 2, 2b - 2, 0, 2b - 1, \forall i = 1, 2, ..., w - 1. \)

**Proof.** Choose \( 0 \leq c_i < m; i = 1, 2, ..., r = 3w \), such that \( b^i \equiv c_i \mod m \).

By Lemma 1 in [1] we have \( a_i \equiv -m^{-1}c_i \mod b; i = 1, 2, ..., 3w \). Now since \( 1 + b^w + b^{2w} \equiv 0 \mod m \) and \( gcd(m,b) = 1 \) then \( c_i + c_{w+i} + c_{2w+i} \equiv 0 \mod m \). Since \( 1 \leq c_i \leq m - 1 \) for all \( i \) then \( c_i + c_{w+i} + c_{2w+i} = m \) or \( 2m \). Thus \( a_i + a_{w+i} + a_{2w+i} \equiv -m^{-1}(c_i + c_{w+i} + c_{2w+i}) \equiv -1 \) or \( -2 \mod b \).

Therefore \( a_i + a_{w+i} + a_{2w+i} \equiv b - 1 \mod b \) and hence \( a_i + a_{w+i} + a_{2w+i} = b - 2b - 2, 2b - 2, 0, 2b - 1. \) Since \( c_{3w} = 1 \) and \( c_w + c_{2w} + c_{3w} = m \) or \( 2m \) then \( c_w + c_{2w} + c_{3w} = m \) and hence \( a_w + a_{2w} + a_{3w} \equiv -m^{-1}(m) \equiv -1 \equiv b - 1 \mod b. \)

Now we use our lemma to give a very simple proof of Ginsberg theorem in base \( b \).

**Theorem 1.** Let \( m \) be relatively prime to the base \( b \). Suppose that the period in base \( b \) of the reciprocal of \( m \) has length \( r = 3w \): \( \frac{1}{m} = 0.[a_1a_2...a_{3w}]_b \).

If \( 1 + b^w + b^{2w} \equiv 0 \mod m \), then \( B_1 + B_2 + B_3 = 10^w - 1 \), where \( B_1 = a_1a_2...a_w, B_2 = a_{w+1}a_{w+2}...a_{2w}, \) and \( B_3 = a_{2w+1}a_{2w+2}...a_{3w} \).
Proof. Note that \( m[(a_{3w} + b^w a_{2w} + b^2w a_w) + b(a_{3w-1} + b^w a_{2w-1} + b^{2w} a_{w-1}) + b^2(a_{3w-2} + b^w a_{2w-2} + b^{2w} a_{w-2}) + \ldots + b^{w-1}(a_{2w+1} + b^w a_{w+1} + b^{2w} a_1)] = b^{3w} - 1 = (b^w - 1)(1 + b^w + b^{2w}) \). Let \( c = b^w - 1 \). Then mod \( c \) we have \( b^w \equiv b^{2w} \equiv 1 \) and since \( 1 + b^w + b^{2w} \equiv 0 \mod m \) then \( (a_{3w} + a_{2w} + a_w) + b(a_{3w-1} + a_{2w-1} + a_{w-1}) + b^2(a_{3w-2} + a_{2w-2} + a_{w-2}) + \ldots + b^{w-1}(a_{2w+1} + a_{w+1} + a_1) \equiv 0 \mod c \). So, \( B_1 + B_2 + B_3 \equiv 0 \mod c \). That is, \( B_1 + B_2 + B_3 = k(b^w - 1) \) for some \( k \in \mathbb{N} \). Using the previous lemma we see that \( B_1 + B_2 + B_3 \leq (2b - 1) + b(2b - 1) + b^2(2b - 1) + \ldots + b^{w-1}(2b - 1) < 3c \). Now assume, on the contrary, that \( B_1 + B_2 + B_3 \neq b^w - 1 \). Then we would have \( B_1 + B_2 + B_3 = 2c \equiv b - 2 \mod b \). However, by the previous lemma, \( B_1 + B_2 + B_3 \equiv a_w + a_{2w} + a_{3w} \equiv b - 1 \mod b \), a contradiction. Therefore \( B_1 + B_2 + B_3 = c = b^w - 1 \), a string of \((b - 1)\)'s.

**Corollary 1.** [3] Suppose that the period of the reciprocal of a prime \( p \geq 5 \) has length \( r = 3w: \frac{1}{p} = 0.a_1a_2...a_{3w} \).

Then \( a_1a_2...a_w + a_{w+1}a_{w+2}...a_{2w} + a_{2w+1}a_{2w+2}...a_{3w} \) is a string of 9’s.

**Proof.** Since \( r = 3w \) is the order of 10 mod \( p \) then \( 10^{3w} - 1 = (10^w - 1)(1 + 10^w + 10^{2w}) \equiv 0 \mod p \) and \( 10^w \) is not congruent to 1 mod \( p \). So \( 1+10^w+10^{2w} \equiv 0 \mod p \). The result now follows immediately from the above theorem. 

**References**


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