Differential Equation of Generalized

$K$-Mittag-Leffler Function

Kuldeep Singh Gehlot

Government Bangur P.G. College, Pali
Pali-Marwar, Rajasthan, India-306401

drksgehot@rediffmail.com

Abstract

In this paper we introduce a homogeneous linear differential equation whose one of the solution is the Generalized K-Mittag-Leffler function and deduce this differential equation for earlier defined different Mittag-Leffler functions.

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1 Introduction

In [1] the authors introduce the generalized K-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C \setminus kZ^-,\tag{1}$$

where $(x)_{n,k}$ is the K-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k), x \in C \setminus kZ^-, k \in R, n \in N^+.\tag{2}$$

The integral form of the generalized K-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t}{k}}dt, x \in C \setminus kZ^-, k \in R, Re(x) > 0,\tag{3}$$
from which it follows easily that

\[ \Gamma_k(x) = k^{\frac{x}{k}} \Gamma_k \left( \frac{x}{k} \right). \]  

(4)

and

\[ (\gamma)_{nq,k} = (k)^{nq} \left( \frac{\gamma}{k} \right)_{nq}. \]  

(5)

The Generalized K- Mittag-Leffler function, introduced by [3], as

**Definition 1:** Let \( k \in \mathbb{R}; \alpha, \beta, \gamma \in \mathbb{C} \); \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0 \) and \( q \in (0, 1) \cup \mathbb{N} \), the Generalized K- Mittag-Leffler function denoted by \( GE_{\gamma,q}^{\alpha,\beta}(z) \) and defined as,

\[ GE_{\gamma,q}^{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{\Gamma_k(n\alpha + \beta)(n)!}, \]  

(6)

where \( (\gamma)_{nq,k} \) is the K- pochhammer symbol given by equation (2) and \( \Gamma_k(x) \) is the K-gamma function given by equation (3).

The generalized Pochhammer symbol(cf. [7], page 22),

\[ (\gamma)_n = \frac{\Gamma(\gamma + nq)}{\Gamma(\gamma)} = q^m \prod_{r=1}^{q} \left( \frac{\gamma + r - 1}{q} \right)_n, \text{ if } q \in \mathbb{N}. \]  

(7)

by using (4), (5) and (7) in equation (6) and choosing \( \alpha = km; m, q \in \mathbb{N} \) and \( k \in \mathbb{R} \), we obtain(cf. [4], page 49),

\[ GE_{\gamma,q}^{\alpha,\beta}(z) = \frac{k^{\frac{1}{k}} \frac{q}{k}}{\Gamma_k(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i)_n \prod_{j=1}^{m} (b_j)_n(n!) (A z)^n. \]  

(8)

Where

\[ a_i = \left( \frac{\frac{\gamma}{k} + i - 1}{q} \right), b_j = \left( \frac{\frac{\beta}{k} + j - 1}{m} \right) \text{ and } A = \frac{q^m k^q}{m^m k^m}. \]  

(9)

**2 Main result**

In this section we introduce a linear homogeneous differential equation known as Generalized K -Mittag-Leffler differential equation. One of its solution is Generalized K -Mittag-Leffler function (8). Finally we deduce this differential equation whose one of the solution is earlier known different Mittag-Leffler functions.
Theorem 1. The Generalized K-Mittag-Leffler differential equation is defined as,

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) - A z \prod_{i=1}^{q} (\theta + a_i) W = 0, q \leq m + 1. \]  

(10)

When no \( b_j \) is a negative integer or zero and no two \( b_j \)'s differ by an integer or zero, then the solution is

\[ W = \sum_{r=0}^{m} C_r W_r. \]  

(11)

Where \( C_r \) is arbitrary constant, and

\[ W_0 = G_{\gamma,q}^{\gamma,q} k_{km,\beta}(z). \]  

(12)

and for \( r = 1, 2, 3, \ldots, m \),

\[ W_r = \sum_{n=0}^{\infty} \prod_{r=1}^{q} (a_i - b_r + 1)n \prod_{j=1}^{m} (b_j - b_r + 1)_{n}(2 - b_r)_n (Az)^{n+b_r}. \]  

(13)

Where \( \theta = z \frac{d}{dz} \) and \( a_i, b_j \) and \( A \) are given by (9).

Proof. Whenever, in addition to the above restrictions, no \( b_j \) is a positive integer, then the linear combination (11) is the general solution of equation (10) around \( z = 0 \). Note also that if \( q \leq m \), then the series for \( W_r; r = 0, 1, 2, \ldots, m \), converge for all finite \( z \) and that for \( q = m + 1 \), the series for \( W_r \) converge for \( | z | < 1 \).

First we will verify that \( W_0 \), satisfies equation (10).

Since \( \theta (Az)^n = n (Az)^n \) and using (8), it follows that

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_0 = \kappa^{1-\beta} \frac{z}{\Gamma(\frac{\beta}{k})} \sum_{n=0}^{\infty} n \prod_{j=1}^{m} (n + b_j - 1) \prod_{i=1}^{q} (a_i)_n \prod_{j=1}^{m} (b_j)_n (n!) (Az)^n, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_0 = \kappa^{1-\beta} \frac{z}{\Gamma(\frac{\beta}{k})} \sum_{n=1}^{\infty} \prod_{i=1}^{q} (a_i)_n \prod_{j=1}^{m} (b_j)_{n-1}(n-1)! (Az)^n, \]

now we replace \( n \) by \( n + 1 \), we have

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_0 = \kappa^{1-\beta} \frac{z}{\Gamma(\frac{\beta}{k})} \sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i)_{n+1} \prod_{j=1}^{m} (b_j)_n (n!) (Az)^{n+1}, \]
\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_0 = k^{1-\frac{q}{k}} \frac{\sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i)_n \prod_{i=1}^{q} (n + a_i) (Az)^{n+1}}{\prod_{j=1}^{m} (b_j)_n(n)!}, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_0 = Az \prod_{i=1}^{q} (\theta + a_i) W_0, \]

Thus we have shown that \( W_0 = GE_{k, km, \beta}^{\gamma, q}(z) \) is a solution of the differential equation (10).

Now we will verify that \( W_r, r = 1, 2, ..., m, \) satisfies equation (10).

From (13), we get immediately

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = \sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i - b_r + 1)_n (Az)^{n+1-b_r}, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = \sum_{n=1}^{\infty} \prod_{j=1}^{m} (b_j - b_r + 1)_n (2 - b_r)_n (Az)^{n+1-b_r}, \]

now we replace \( n \) by \( n + 1 \), we have

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = Az \sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i - b_r + 1)_{n+1} (Az)^{n+1-b_r}, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = Az \sum_{n=0}^{\infty} \prod_{j=1}^{m} (b_j - b_r + 1)_n (2 - b_r)_n (Az)^{n+1-b_r}, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = Az \sum_{n=0}^{\infty} \prod_{i=1}^{q} (a_i - b_r + 1)_n \prod_{i=1}^{q} (\theta + a_i) (2 - b_r)_n (Az)^{n+1-b_r}, \]

\[ \theta \prod_{j=1}^{m} (\theta + b_j - 1) W_r = Az \prod_{i=1}^{q} (\theta + a_i) W_r. \]

Thus we have shown that, \( W_r, r = 1, 2, ..., m \) is the solutions of the differential equation (10).
**Particular Cases:** For some particular values of the parameters $q, k, \alpha, \beta$ and $\gamma$, we can obtain certain differential equations for different Mittag-Leffler functions, here we have chosen $\alpha = km; m, q \in N$ and $k \in R$.

[A] Put $q = 1$ in (10), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \quad (14)$$

Here $a_1 = \frac{\gamma}{k}, b_j = \left(\frac{\beta+j-1}{m}\right)$ and $A = \frac{k^{1-m}}{m^m}$.

Equation (14), is the differential equation of K-Mittag-Leffler function $W_0 = E_{k,km,\beta}^\gamma(z)$, defined by [2].

[B] Put $k = 1$ in (10), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az \prod_{i=1}^{q}(\theta + a_i)]W = 0, q \leq m + 1. \quad (15)$$

Here $a_i = \left(\frac{\gamma+i-1}{q}\right), b_j = \left(\frac{\beta+j-1}{m}\right)$ and $A = \frac{q^q}{m^m}$.

Equation (15), is the differential equation of Mittag-Leffler function $W_0 = E_{m,\beta}^\gamma(z)$, defined by [8].

[C] Put $k = 1, q = 1$ in (10), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \quad (16)$$

Here $a_1 = \gamma, b_j = \left(\frac{\beta+j-1}{m}\right)$ and $A = \frac{1}{m^m}$.

Equation (16), is the differential equation of Mittag-Leffler function $W_0 = E_{m,\beta}^\gamma(z)$, defined by [6].

[D] Put $k = 1, q = 1$ and $\gamma = 1$ in (10), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \quad (17)$$

Here $a_1 = 1, b_j = \left(\frac{\beta+j-1}{m}\right)$ and $A = \frac{1}{m^m}$.

Equation (17), is the differential equation of Mittag-Leffler function $W_0 = E_{m,\beta}^1(z)$, defined by [6].
$E_{m,\beta}(z)$, defined by [9].

[\textbf{E}] Put $k = 1$, $q = 1$, $\gamma = 1$ and $\beta = 1$ in (10), we have the differential equation.

$$[\theta \prod_{j=1}^{m}(\theta + b_j - 1) - Az(\theta + a_1)]W = 0. \quad (18)$$

Here $a_1 = 1$, $b_j = \left(\frac{j}{m}\right)$ and $A = \frac{1}{m^m}$.

Equation (18), is the differential equation of Mittag-Leffler function $W_0 = E_m(z)$, defined by [5].

Reference


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