Some Properties of a Semi-symmetric Non-metric Connection on a Sasakian Manifold

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Abstract

The object of the present paper is to study some properties of semi-symmetric non-metric connection on a Sasakian manifold.

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1 Introduction

Let \( (M_n, g) \) be a Riemannian manifold of dimension \( n \). A linear connection \( \nabla \) in \( (M_n, g) \), whose torsion tensor \( T \) of type \((1, 2)\) is defined as

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]
\]

for arbitrary vector fields \( X \) and \( Y \), is said to be torsion free or symmetric if \( T \) vanishes, otherwise it is non-symmetric. If the connection \( \nabla \) satisfies \( \nabla g = 0 \) in \( (M_n, g) \), then it
is called metric connection otherwise it is non-metric. In [1], Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a differentiable manifold. Hayden [2] introduced the idea of semi-symmetric linear connection with non zero torsion on a Riemanian manifold. The idea of semi-symmetric metric connection on Riemannian manifold was introduced by Yano [3]. He proved that a Riemannian manifold with respect to the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. This result was generalized for vanishing Ricci tensor of the semi-symmetric metric connection by T. Imai ([4], [5]). Various properties of such connection have studied by many geometers. Agashe and Chaffe [7] defined and studied a semi-symmetric non-metric connection in a Riemannian manifold. This was further developed by Agashe and Chaffe [8], Prasad [9], De and Kamilya [12], Tripathi and Kakkar [14], Pandey and Ojha [10], Chaturvedi and Pandey [15] and other geometers. Sengupta, De and Binh [18], De and Sengupta [11] defined new type of semi-symmetric non-metric connections on a Riemannian manifold and studied some geometrical properties with respect to such connections. In this connection, the properties of semi-symmetric non-metric connections have studied by Özgür [25], Ahmad and Özgür [26], Özgür and Sular [28], Kumar and Chaubey [20], Dubey, Chaubey and Ojha [19] and many other geometers. In 2008, Tripathi introduced the generalized form of a new connection in Riemannian manifold [24]. In [16, 17], Chaubey defined semi-symmetric non-metric connections on an almost contact metric manifold and studied its different geometrical properties. Some properties of such connection [17] have further studied by Jaiswal and Ojha [21], Chaubey and Ojha [22] and Chaubey [17]. In the present paper, we studied the properties of such connection in Sasakian manifolds.

The present paper is organized as follows. Section 2 is preliminaries in which some basic definitions are given. The next section deals with the semi-symmetric non-metric connection. Section 4 is concerned with the projective and Riemannian curvature tensors equipped with semi-symmetric non-metric connection and proved that for a Sasakian manifold with the Riemannian connection is $\xi$-projectively flat if and only if it is also $\xi$-projectively flat with respect to the semi-symmetric non-metric connection. It has also shown that if a Sasakian manifold is $\phi$-projectively flat, then the manifold is a $\eta$-Einstein manifold. In the last we have shown the necessary and sufficient condition for a Sasakian manifold to be locally $\phi$-symmetric with respect to the semi-symmetric non-metric connection.

2 Preliminaries

An $n$-dimensional Riemannian manifold $(M_n, g)$ of class $C^\infty$ with a 1-form $\eta$, the associated vector field $\xi$ and a $(1, 1)$ tensor field $\phi$ satisfying

$$\phi^2 X + X = \eta(X)\xi, \quad (2)$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad (3)$$
for arbitrary vector field $X$, is called an almost contact manifold. The system $(\phi, \xi, \eta)$ is called an almost contact structure to $M_n$ [6]. If the associated Riemannian metric $g$ in $M_n$ satisfy

$$g(\phi X, \phi Y) = g(X, Y) - \eta(Y)\eta(Y),$$

(4)

for arbitrary vector fields $X$, $Y$ in $M_n$, then $(M_n, g)$ is said to be an almost contact metric manifold. Putting $\xi$ for $X$ in (4) and using (3), we obtain

$$g(\xi, Y) = \eta(Y).$$

(5)

Also,

$$\varphi(X, Y) = g(\phi X, Y)$$

(6)

gives

$$\varphi(X, Y) + \varphi(Y, X) = 0.$$ 

(7)

An almost contact metric manifold $M_n$ is said to be a Sasakian manifold [6] if it satisfies the following tensorial relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

(8)

for arbitrary vector fields $X$ and $Y$ on $M_n$. Here $\nabla$ denotes the Levi-Civita connection of the metric $g$. A normal contact metric manifold of dimension $n$ greater than or equal to three is called a Sasakian manifold.

Let $R$ be the curvature tensor of type $(1, 3)$ and $S$ is the Ricci tensor of type $(0, 2)$ with respect to the Levi-Civita connection $\nabla$, then the following relations hold in a Sasakian manifold for any arbitrary vector fields $X$ and $Y$:

$$\nabla_X \xi = -\phi X,$$

(9)

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

(10)

$$(\nabla X \eta)(Y) = g(X, \phi Y),$$

(11)

$$R(\xi, X)Y = (\nabla X \phi)(Y),$$

(12)

$$S(\phi X, \phi Y) = S(X, Y) - (n - 1)\eta(X)\eta(Y),$$

(13)

$$S(X, \xi) = (n - 1)\eta(X).$$

(14)

The projective curvature tensor is an important tensor having one-one correspondence between each coordinate neighbourhood of an n-dimensional Riemannian manifold and a domain of Euclidean space such that there is one-one correspondence between geodesics of Riemannian manifold with straight line in Euclidean space. A manifold of dimension $n$, $(n \geq 3)$ is projectively flat if the tensorial relation of projective curvature tensor vanishes. The projective curvature tensor is given by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n - 1}\{S(Y, Z)X - S(X, Z)Y\},$$

(15)

for $X$, $Y$ and $Z \in T(M_n)$, where $T(M_n)$ denotes the tangent space at each point of the manifold $M_n$. The manifold $M_n$ is projectively flat if and only if the manifold is of constant curvature.
Definition 2.1 A Sasakian manifold $M_n$ is said to be $\xi$–projectively flat if the condition $P(X,Y) \xi = 0$ holds on $M_n$ for arbitrary vector fields $X$ and $Y$.

Definition 2.2 A Sasakian manifold $M_n$ is said to be $\phi$–projectively flat if

$$\phi^2(P(\phi X, \phi Y)\phi Z) = 0,$$

for vector fields $X, Y, Z \in T(M_n)$.

Analogous to the above definitions, we define the following definitions:

Definition 2.3 A Sasakian manifold $M_n$ equipped with semi-symmetric non-metric connection $\tilde{\nabla}$ is said to be $\xi$–projectively flat with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ if the condition $\tilde{P}(X,Y) \xi = 0$ holds on $M_n$ for arbitrary vector fields $X$ and $Y$.

Definition 2.4 A Sasakian manifold $M_n$ equipped with semi-symmetric non-metric connection $\tilde{\nabla}$ is said to be $\phi$–projectively flat with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ if the condition

$$\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0$$

holds for vector fields $X, Y, Z \in T(M_n)$.

3 Semi-symmetric non-metric connection on Sasakian manifold

A linear connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \phi(X,Y)\xi$$

is called a semi-symmetric non-metric connection [17] if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ is defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$

for any vector fields $X, Y$ satisfies

$$\tilde{T}(X,Y) = 2\phi(X,Y)\xi$$

and

$$(\tilde{\nabla}_X g)(Y,Z) = -\eta(Y)\phi(X,Z) - \eta(Z)\phi(X,Y)$$

for arbitrary vector fields $X, Y$ and $Z$. 
It is well known [21, 22] that
\[(\tilde{\nabla}_X \phi)(Y, Z) = (\nabla_X \phi)(Y, Z),\] (22)
\[(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - g(%X, Y)%.\] (23)

Interchanging \(X\) and \(Y\) in (23), we have
\[(\tilde{\nabla}_Y \eta)(X) = (\nabla_Y \eta)(X) - g(%Y, X%).\] (24)

Subtracting (24) from (23) and using (6), we get
\[(\tilde{\nabla}_X \eta)(Y) - (\tilde{\nabla}_Y \eta)(X) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) - 2g(%X, Y%).\] (25)

An almost contact metric manifold \(M_n\) is said to be an almost Sasakian manifold [27] with respect to the Levi-Civita connection \(\nabla\) if and only if
\[\phi(X, Y) = \frac{1}{2}[(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)].\] (26)

In view of (25) and (26), we state the following theorem:

**Theorem 3.1** Let \(M_n\) be an almost contact metric manifold equipped with a semi-symmetric non-metric connection \(\tilde{\nabla}\), then \(M_n\) is an almost Sasakian manifold if and only if
\[(\tilde{\nabla}_X \eta)(Y) = (\tilde{\nabla}_Y \eta)(X).\] (27)

4 Curvature tensor of a Sasakian manifold with respect to semi-symmetric non-metric connection

A relation between the curvature tensors \(R\) and \(\tilde{R}\) of the Levi-Civita connection \(\nabla\) and semi-symmetric non-metric connection \(\tilde{\nabla}\) respectively is given by [17, 22]
\[\tilde{R}(X, Y, Z) = R(X, Y, Z) + g(%Y, Z)\nabla_X \xi - g(%X, Z)\nabla_Y \xi + g((\nabla_X \phi)(Y) - (\nabla_Y \phi)(X), Z)\xi,\] (28)

for arbitrary vectors fields \(X\), \(Y\) and \(Z\). In consequence of (8) and (9), (28) becomes
\[\tilde{R}(X, Y)Z = R(X, Y)Z - g(%Y, Z)\phi X + g(%X, Z)\phi Y - \eta(Y)g(X, Z)\xi + \eta(X)g(Y, Z)\xi.\] (29)

Contracting (29) along \(X\), we get
\[\tilde{S}(Y, Z) = S(Y, Z)\] (30)
which give

\[ \tilde{Q}Y = QY \]  \hspace{1cm} (31)

and

\[ \tilde{r} = r, \]  \hspace{1cm} (32)

where \( \tilde{S} \) denotes the Ricci tensor with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) and \( \tilde{r} \); \( r \) denote the scalar curvatures with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) and the Levi-Civita connection \( \nabla \) respectively. Thus with the help of (30) and (32), we state the following corollaries:

**Corollary 1** If \( M_n \) is an \( n \)-dimensional Sasakian manifold equipped with a semi-symmetric non-metric connection \( \tilde{\nabla} \) then the Ricci tensors with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) and Levi-Civita connection \( \nabla \) are invariant.

**Corollary 2** If an \( n \)-dimensional Sasakian manifold \( M_n \) admits a semi-symmetric non-metric connection \( \tilde{\nabla} \), then the scalar curvatures with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) and Levi-Civita connection \( \nabla \) coincide.

Analogous to (15), the projective curvature tensor \( \tilde{P} \) with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) is defined as

\[ \tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y], \]  \hspace{1cm} (33)

for arbitrary vector fields \( X, Y, Z \).

The generalised projective curvature tensor \( \bar{P} \) [27] with respect to the Levi-Civita connection \( \nabla \) is defined as

\[ \bar{P}(X,Y)Z = R(X,Y)Z + \frac{1}{n+1}\{S(X,Y) - S(Y,X)\}Z \]
\[ + \frac{1}{(n^2 - 1)}\{nS(X,Z) + S(Z,X)\}Y - \{nS(Y,Z) + S(Z,Y)\}X, \]  \hspace{1cm} (34)

for arbitrary vector fields \( X, Y, Z \).

Analogous to (34), we define the generalized projective curvature tensor \( \tilde{P} \) with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \) as

\[ \tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n+1}\{\tilde{S}(X,Y) - \tilde{S}(Y,X)\}Z \]
\[ + \frac{1}{(n^2 - 1)}\{n\tilde{S}(X,Z) + \tilde{S}(Z,X)\}Y - \{n\tilde{S}(Y,Z) + \tilde{S}(Z,Y)\}X. \]  \hspace{1cm} (35)

In consequence of (29), (30), (33) and the symmetric properties of Ricci tensor, the above relation becomes

\[ \tilde{P}(X,Y)Z = \tilde{P}(X,Y)Z, \]  \hspace{1cm} (36)
where $\tilde{P}$ denotes the projective curvature tensor with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ and $\tilde{Q}$ is the Ricci operator with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$, defined as

$$\tilde{S}(X,Y) \overset{\text{def}}{=} g(\tilde{Q}X,Y),$$

(37)

for arbitrary vector fields $X$ and $Y$. With the help of (15), (29), (30), (33) and the symmetric properties of Ricci tensor, it follows that

$$\tilde{P}(X,Y)Z = P(X,Y)Z - g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y - \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi.$$  

(38)

Replacing $Z$ by $\xi$ in (38) and then using (3) and (5), we find

$$\tilde{P}(X,Y)\xi = P(X,Y)\xi.$$  

(39)

Thus in consequence of definition (2) and equation (39), we state the following theorem:

**Theorem 4.1** A Sasakian manifold $M_n$ is said to be $\xi$–projectively flat with respect to the Levi-Civita connection $\nabla$ if and only if it is $\xi$–projectively flat with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$.

Let us suppose that the Sasakian manifold $M_n$ is projectively flat with respect to the semi symmetric non-metric connection $\tilde{\nabla}$, then we have

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0$$

(40)

for all $X, Y, Z \in T(M_n)$. With the help of equations (33) and (40), we find

$$g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi W) = \frac{1}{n-1}[\tilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) - \tilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)].$$

(41)

We suppose that the set $\{e_1, e_2, e_3, \ldots, e_{n-1}, \xi\}$ is an orthonormal basis of an $n$–dimensional Sasakian manifold $M_n$ then the set $\{\phi e_1, \phi e_2, \phi e_3, \ldots, \phi e_{n-1}, \xi\}$ also form the basis of $M_n$. Taking $X = W = e_i$ in the above equation and then taking summation over $i$, $1 \leq i \leq (n - 1)$, we get

$$\sum_{i=1}^{n-1} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) = \frac{1}{n-1} \sum_{i=1}^{n-1} [\tilde{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - \tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i)].$$

(42)

In view of (3), (5), (29) and (30), we have

$$\sum_{i=1}^{n-1} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) = \sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) - g(\phi Y,\phi Z)$$

$$= S(Y,Z) - R(\xi,Y,Z,\xi) - (n-1)\eta(Y)\eta(Z) - g(\phi Y,\phi Z)$$

$$= \tilde{S}(Y,Z) - 2g(Y,Z) - (n-3)\eta(Y)\eta(Z),$$

(43)
where
\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi e_j) = n - 1 \] (44)

and
\[ \sum_{i=1}^{n-1} \tilde{S}(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = \tilde{S}(\phi Y, \phi Z). \] (45)

With the help of (42), (43), (44) and (45), it follows that
\[ \tilde{S}(Y, Z) - 2g(Y, Z) - (n - 3)\eta(Y)\eta(Z) = \frac{n - 2}{n - 1} \tilde{S}(\phi Y, \phi Z) \] (46)

In view of (13) and (30), (46) becomes
\[ \tilde{S}(Y, Z) - 2g(Y, Z) - (n - 3)\eta(Y)\eta(Z) = \frac{n - 2}{n - 1} \tilde{S}(Y, Z) - (n - 1)\eta(Y)\eta(Z) \]

which gives
\[ S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \] (47)

where \( a = 2(n - 1) \) and \( b = -(n - 1) \). Thus we state the following theorem:

**Theorem 4.2** A \( \phi \)-projectively flat Sasakian manifold \( M_n \) equipped with a semi-symmetric non-metric connection \( \tilde{\nabla} \) is a \( \eta \)-Einstein manifold.

### 5 Locally \( \phi \)-symmetric Sasakian manifold with respect to the semi-symmetric non-metric connection \( \tilde{\nabla} \)

Let \( M_n \) be an \( n \)-dimensional Sasakian manifold equipped with a semi-symmetric non-metric connection \( \tilde{\nabla} \). From (3), (5), (6), (10), (12) and (18), it follows that

\[
(\tilde{\nabla}_W \tilde{R})(X, Y)Z = (\tilde{\nabla}_W \tilde{R})(X, Y)Z + g(\phi W, \tilde{R}(X, Y)Z)\xi \\
+ g(\phi X, W) \{g(Y, Z)\xi - \eta(Z)Y\} \\
- g(\phi Y, W) \{g(X, Z)\xi - \eta(Z)X\} \\
+ g(\phi Z, W) \{\eta(Y)X - \eta(X)Y\}.
\] (48)

Taking covariant derivative of (29) with respect to the Levi-Civita connection \( \nabla \) along \( W \) and then using (29), we get

\[
(\nabla_W \tilde{R})(X, Y)Z = (\nabla_W \tilde{R})(X, Y)Z - g((\nabla_W \phi)(Y), Z)\phi X - g(\phi Y, Z)(\nabla_W \phi)(X) \\
+ g((\nabla_W \phi)(X), Z)\phi Y + g(\phi X, Z)(\nabla_W \phi)(Y) - (\nabla_W \eta)(Y)g(X, Z)\xi \\
- \eta(Y)g(X, Z)\nabla_W \xi + (\nabla_W \eta)(X)g(Y, Z)\xi + \eta(X)g(Y, Z)\nabla_W \xi.
\] (49)
In consequence of (3), (5), (8), (9) and (11), (49) becomes

\[(\nabla_w \tilde{\nabla})(X, Y)Z = (\nabla_w R)(X, Y)Z - \eta(Z)g(Y, W)\phi X + \eta(Y)g(W, Z)\phi X - g(\phi Y, Z)g(W, X)\xi + \eta(X)g(\phi Y, Z)W + \eta(Z)g(X, W)\phi Y - \eta(X)g(W, Z)\phi Y + g(\phi X, Z)g(Y, W)\xi - \eta(Y)g(\phi X, Z)W - g(\phi Y, W)g(X, Z)\xi - \eta(Y)g(X, Z)\phi W + g(\phi X, W)g(Y, Z)\xi + \eta(X)g(Y, Z)\phi W. \tag{50}\]

From (48) and (50), it follows that

\[(\tilde{\nabla}_w \tilde{\nabla})(X, Y)Z = (\nabla_w R)(X, Y)Z - \eta(Z)g(Y, W)\phi X + \eta(Y)g(W, Z)\phi X - g(\phi Y, Z)g(W, X)\xi + \eta(X)g(\phi Y, Z)W + \eta(Z)g(X, W)\phi Y - \eta(X)g(W, Z)\phi Y + g(\phi X, Z)g(Y, W)\xi - \eta(Y)g(\phi X, Z)W - g(\phi Y, W)g(X, Z)\xi - \eta(Y)g(X, Z)\phi W + g(\phi X, W)g(Y, Z)\xi + \eta(X)g(Y, Z)\phi W + g(\phi X, W)\{g(Y, Z)\xi - \eta(Z)Y\} - g(\phi Y, W)\{g(X, Z)\xi - \eta(Z)X\}. \tag{51}\]

Operating \(\phi^2\) on either sides of (51) and using (2) and (3), we get

\[\phi^2(\tilde{\nabla}_w \tilde{\nabla})(X, Y)Z = \phi^2(\nabla_w R)(X, Y)Z + \eta(Z)g(Y, W)\phi X - \eta(Y)g(W, Z)\phi X - \eta(X)g(\phi Y, Z)W + \eta(X)g(\phi Y, Z)\eta(W)\xi - \eta(Z)g(X, W)\phi Y + \eta(X)g(W, Z)\phi Y + \eta(Y)g(\phi X, Z)W - \eta(Y)g(\phi X, Z)\eta(W)\xi - \eta(X)g(Y, Z)\phi W - g(\phi Z, W)\{\eta(Y)X - \eta(X)Y\} + g(\phi X, W)\{\eta(Z)Y - \eta(Z)\eta(Y)\xi\} + \eta(Y)g(X, Z)\phi W - g(\phi Y, W)\{\eta(Z)X - \eta(Z)\eta(X)\xi\}. \tag{52}\]

If we take the vector fields \(X, Y, Z\) and \(W\) orthogonal to \(\xi\), then (52) reduces to

\[\phi^2(\tilde{\nabla}_w \tilde{\nabla})(X, Y)Z = \phi^2(\nabla_w R)(X, Y)Z.\]

Thus we can state the following theorem:

**Theorem 5.1** The necessary and sufficient condition for a Sasakian manifold to be locally \(\phi\)-symmetric with respect to the Levi-Civita connection \(\nabla\) is that it is locally \(\phi\)-symmetric with respect to the semi-symmetric non-metric connection \(\tilde{\nabla}\).

**References**


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