On the Real Part of the Frobenius Companion Matrix of Polynomials

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Abstract

In this article, we derive an expansion of the characteristic polynomial of the Frobenius companion matrix of a monic polynomial as a linear combination of a type of Chebyshev polynomials.

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1. Introduction

Let

\[ p(z) = z^n + a_n z^{n-1} + \ldots + a_2 z + a_1 \]

be a monic polynomial of degree \( n \geq 3 \), with complex coefficients \( a_1, a_2, \ldots, a_n \).

It is well known that the zeros of \( p \) are exactly the eigenvalues of the Frobenius companion matrix of \( p \) which is given by

\[
C = \begin{bmatrix}
-a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

(1)

See, e.g., [2, p.316]. Let \( R = \frac{1}{2}[C + C^*] \) be the Cartesian real part of \( C \), where \( C^* \) is the Hermitian adjoint of \( C \).

In this paper, we derive an explicit form of the characteristic polynomial of \( R \). The form we obtain is a linear combination of a type of Chebyshev polynomials.

2. Preliminary Results

For an \( n \times n \) Hermitian matrix \( A \), let \( \det A \) stand for the determinant of \( A \), \( \text{adj} A \) be the adjugate (classical adjoint) of \( A \) and \( \lambda_1(A) \leq \ldots \leq \lambda_n(A) \) be the eigenvalues of \( A \).

The following lemma is an immediate consequence of the Courant-Fischer minimax principle, See, e.g., [4, chapter 4].

**Lemma 1.** If \( z \) is any zero of \( p \), then

\[
\lambda_1(R) \leq \text{Re}z \leq \lambda_n(R)
\]
It follows from (1) that $R$ has the partitioned form

$$R = \begin{bmatrix} -\Re a_n & -x^* \\ -x & B \end{bmatrix},$$

(2)

where

$$x = [\frac{1}{2} (\tilde{\alpha}_{n-1} - 1), \frac{1}{2} \tilde{\alpha}_{n-2}, \ldots, \frac{1}{2} \tilde{\alpha}_{2}, \frac{1}{2} \tilde{\alpha}_1]^T$$

and $B$ is the $(n-1) \times (n-1)$ tridiagonal matrix

$$B = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \vdots \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$ 

It is well-known that the eigenvalues of $B$ are $\cos \frac{j\pi}{k+1}$ for $j = 1, 2, \ldots, k$.

### 3. Main Results

Let $T_k$ denote the $k \times k$ tridiagonal matrix

$$T_k = \begin{bmatrix} t & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & t & -\frac{1}{2} & \cdots & \vdots \\ 0 & -\frac{1}{2} & t & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -\frac{1}{2} & t \end{bmatrix}.$$

It is easy to see that $\lambda I - R = \begin{bmatrix} \lambda + \Re a_n & x^* \\ x & T_{n-1} \end{bmatrix}$.

**Lemma 2.** Let $D_k = \det T_k$, then

$$D_k = \frac{\sin(k+1)\theta}{2^k \sin \theta}, \text{ where } \cos \theta = t$$

**Proof.** It is easy to verify that the sequence $D_k$ satisfy the recurrence relation
\[ D_k = tD_{k-1} - \frac{1}{4} D_{k-2} \quad \text{with} \quad D_1 = t \quad \text{and} \quad D_0 = 1. \quad (3) \]

which has the solution \[ D_k = \frac{\sin(k+1)\theta}{2^t \sin \theta}, \quad \text{with} \quad \cos \theta = t. \quad \square \]

Observe that the relation (3) identifies \( D_k \) as a type of Chebyshev polynomials.

Now we state and prove our main result.

**Theorem 1.** Set \( a_n = \frac{1}{2} \), then the characteristic polynomial of \( R \) can be written as

\[ \det(\lambda I - \text{Re } C(p)) = \sum_{k=0}^{n} \frac{1}{2^{n-k}} \left[ 2\text{Re}(a_{k+1}) - \sum_{m=1}^{n-k-1} \sum_{r=0}^{k} \text{Re}(a_{r+m-n}d_{r+m,m}) \right] D_k, \quad (4) \]

a linear combination of the polynomials \( D_k \).

**Proof.** It can be shown [4, p. 175] that the characteristic polynomial of \( R \) can be written as

\[ \det(\lambda I - R) = (\lambda + \text{Re } a_n) \det T_{n-1} - x^*(\text{adj} T_{n-1})x. \quad (5) \]

With a little effort one can check that \( \text{adj}(T_{n-1}) = [d_{ij}]_{(n-1)\times(n-1)} \) where

\[ d_{ij} = \begin{cases} \frac{1}{2^t} D_{n-i-1} D_{j-1}, & i \geq j \\ \frac{1}{2^t} D_{n-j-1} D_{i-1}, & i < j. \end{cases} \]

Thus (5) can be rewritten as

\[ \det(\lambda I - R) = (\lambda + \text{Re } a_n) D_{n-1} - \frac{1}{2^t} \sum_{i \geq j} \frac{2}{2^t} \text{Re}(x_i \overline{x_j}) D_{n-i-1} D_{j-1} - \frac{1}{2^t} \sum_{i \geq j} x_i \overline{x_j} D_{n-i-1} D_{j-1} \quad (6) \]

Using the linearization
\[ D_k D_l = \frac{1}{2^{k+l}} \sin(k+1)\theta \sin(l+1)\theta \cdot \sin\theta = \frac{1}{2^{k+l}} \min[l,k] \sum_{m=0}^{\min[l,k]} \frac{\sin(k+l-2m+1)\theta}{\sin\theta} = \sum_{m=0}^{\min[l,k]} \frac{1}{2^{2m}} D_{k+l-2m} \]

we obtain

\[ \det(\lambda I_n - R) = (\lambda + \text{Re} a_n)D_{n-1} - \sum_{j=1}^{n-1} \frac{2}{2^j} \text{Re}(x_j, \bar{x}_j) \sum_{m=0}^{\min[n-j-1,j-1]} \frac{1}{2^{2m}} D_{n-j-2m-2} \]

(7)

\[ -\sum_{j=1}^{n-1} (x_j, \bar{x}_j) \sum_{m=0}^{\min[n-j-1,j-1]} \frac{1}{2^{2m}} D_{n-2m-2} \]

(8)

It can be shown that the coefficient of \( D_{n-s} \) for \( s = 3, 5, \ldots, 2\left\lfloor \frac{n+1}{2} \right\rfloor + 1 \) in the term (7) is

\[ -\frac{1}{2^{s-2}} \sum_{m=1}^{(s-1)/2} 2^{s-2} \sum_{j=m}^{n-s+m} \text{Re}(x_{j+s-2m}, \bar{x}_j) \]

\[ = -\frac{1}{2^{s-2}} \sum_{m=1}^{(s-1)/2} \sum_{j=m}^{n-s+m} \text{Re}(x_{j+s-2m}, \bar{x}_j) \]

and that the coefficient of \( D_{n-s} \) for \( s = 2, 4, 6, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor \) in the terms (7) and (8) are

\[ -\frac{1}{2^{s-2}} \sum_{m=1}^{(s/2)-1} 2^{s-2} \sum_{j=m}^{n-s+m} \text{Re}(x_{j+s-2m}, \bar{x}_j) \]

and

\[ \frac{1}{2^{s-2}} \sum_{i=s/2}^{n-s/2} x_i \bar{x}_i \]

respectively, which imply that the coefficient of \( D_{n-s} \) for \( s = 2, 4, 6, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor \) in (6) is

\[ -\frac{1}{2^{s-2}} \sum_{m=1}^{n-s} \sum_{r=0}^{n-s} \text{Re}(x_{r+s-m}, \bar{x}_{r+m}) \]

Thus, the expansion (6) becomes

\[ \det(\lambda I_n - \text{Re}(p)) = (t + \text{Re} a_n)D_{n-1} - \left[ \sum_{j=1}^{n-1} \frac{2}{2^{s-2}} \sum_{m=1}^{(s-1)/2} \sum_{j=m}^{n-s+m} \text{Re}(x_{j+s-2m}, \bar{x}_j) \right] D_{n-s} \]

(9)
Substituting \( x_{r+s-m} \overline{x}_{r+m} = \frac{1}{2} a_{r+s-m} \overline{a}_{r+m} \) except when:

1) \( r + s - m = n - 1 \Rightarrow m = 1, r = n - s \) and \( s \neq 2 \Rightarrow x_{r+s-m, r+m} = \frac{1}{2} a_{r-s+1} (a_{n-1} - 1) \); 
2) \( r + m = n - 1 \Rightarrow m = s - 1 \) and \( r = n - s \Rightarrow x_{r+s-m, r+m} = \frac{1}{2} a_{n-s+1} (\overline{a}_{s+1} - 1) \); 
3) \( r + s - m = n - 1 = r + m \Rightarrow s = 2 \) and so \( r = n - 2 \Rightarrow \) the coefficient of \( D_{m-2} \)

is \(- \frac{1}{2} (a_{n-1} - 1)(\overline{a}_{n-1} - 1)\),

then using (3), we obtain that (9) can be written as

\[
\det(a_n - \Re C(p)) = D_n + (\Re a_n) D_{n-1} + \sum_{s=2}^n \frac{1}{2s} \left[ 2 \Re(a_{n-s+1}) - \sum_{m=1}^{s-1} \sum_{r=0}^{n-s} \Re(a_{r+s-m} \overline{a}_{r+m}) \right] D_{n-s}.
\]

Finally if we set \( a_{n+1} = \frac{1}{2} \), then (10) can be written as (4). □

4. Remarks

(1) By similar derivations, one can obtain an analogous expression for the characteristic polynomial of the Cartesian imaginary part of \( C \),

\[ \Im C = \frac{1}{2s} [C - C^*] \], noticing that \( \Im C = \Re [-iC] \).

(2) As a consequence of related results that can be found in [3] and [5], the rectangle \([\lambda_{\min} (\Re C), \lambda_{\max} (\Re C)] \times [\lambda_{\min} (\Im C), \lambda_{\max} (\Im C)]\) contains all the zeros of \( p \). Related rectangles have been also described in [6].

(3) With the results above, consideration may be given to numerical methods for finding zeros of polynomials expressed in a Chebyshev basis. See, e.g., [1] and [2].

References

Frobenius companion matrix of polynomials


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