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A Beale-Kato-Majda Regularity Criteria to the 2D Viscous MHD Equations in BMO Space

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Abstract

In this paper we investigate the Cauchy problem for $2D$ viscous MHD equations with incompressible conditions and establish a Beale-Kato-Majda regularity criteria in term of the velocity vector in the homogeneous BMO space.

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1 Introduction

This paper is devoted to the study of the Beale-Kato-Majda regularity criteria of smooth solutions to the Cauchy problem for $2D$ viscous MHD equations with incompressible conditions

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = b \cdot \nabla b, \tag{1}$$

$$b_t + u \cdot \nabla b - \eta \triangle b = b \cdot \nabla u, \tag{2}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \tag{3}$$

$$t = 0 : u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \tag{4}$$

where $u = (u_1, u_2) \in R^2$ is the velocity field, $b = (b_1, b_2) \in R^2$ is the magnetic field, $p \in R$ is the scalar pressure, $\nu > 0$ is the viscosity, $\eta > 0$ is called the magnetic diffusivity, u_0 and b_0 are the given initial velocity field and initial magnetic field with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$ in the sense of distribution.

Recently, the global well-posedness and asymptotic behavior of the $3D$ MHD equations have drawn a lot of attentions of mathematicians, especially the issue of regularity criteria for the strong and weak solutions. For example, Cao and Wu [2] gave two regularity criteria of strong solutions in Sobolev spaces, He and Xin [5] derived a regularity criterion with the velocity field for the weak solutions of $3D$ MHD equations, Gala [6] investigated the blow-up criteria of $3D$ MHD equations in multiplier spaces. For more interesting results, we may refer to Chen, Miao and Zhang [3], Fan, Jiang, Nakamura and Zhou [4], Jia and Zhou [7], Zhou [9], etc.

The purpose of this paper is to establish the Beale-Kato-Majda regularity criteria of smooth solutions on the velocity field or on the gradient of velocity field in terms of BMO space. The novelty of the proof is to use the incompressible conditions and construct auxiliary terms such as $\int_{R^2} \nabla b \cdot \nabla b \nabla u dx = \int_{R^2} [\nabla(b \cdot \nabla b) - b \cdot \nabla\nabla b] \nabla u dx = -\int_{R^2} u \cdot \nabla\nabla b \nabla b dx - \int_{R^2} u \cdot \nabla b \nabla^2 b dx$, then by virtue of the interpolation inequality to conclude our result.

The paper is organized as follows. We first state some preliminary on functional settings and some important inequalities in Section 2. Theorem 3.1 will be proved in Section 3.

2 Preliminary

Throughout this paper we use the following usual notations.
$L^p(R^3)$ denotes the Lebegue space, $H^m(R^3)$ denotes the standard Sobolev space. BMO denotes the space of bounded mean oscillations. $\mathcal{S}(R^n)$ be the Schwartz class of rapidly decreasing functions.

The Fourier transformation of $f \in \mathcal{S}(R^n)$ is defined as $\mathcal{F}f = \hat{f}(\xi) = \int_{R^n} e^{-ix\cdot\xi} f(x)dx$ and the inverse Fourier transformation of $g \in \mathcal{S}(R^n)$ is defined as $\mathcal{F}^{-1}g = \breve{g}(x) = \int_{R^n} e^{ix\cdot\xi} g(\xi)d\xi$.

Definition 2.1 *The frequency localization operator is defined as*

$$\Delta_k u = \int_{R^n} \breve{\phi}(y) u(x - 2^{-q}y)dy. \tag{5}$$

Definition 2.2 *BMO denotes the homogeneous bounded mean oscillation space which defined as*

$$\|f\|_{BMO} = \sup_{x\in R^n, R>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(y) - \frac{1}{B_r(y)} \int_{B_r(y)} f(z)dz \right| dy. \tag{6}$$

Definition 2.3 *(Triebel-Lizorkin space $\dot{F}^s_{p,q}$) The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ defined as the set of tempered distributions u such that*

$$\|u\|_{\dot{F}^s_{p,q}} = \Big\| \Big(\sum_{k\in Z} 2^{sqk} |\Delta_k u|^q \Big)^{1/q} \Big\|_{L^p} < +\infty. \tag{7}$$

Moreover, when $s = 0,\ p = \infty,\ q = 2$, $\dot{F}^0_{\infty,2} = BMO$.

Lemma 2.4 *The following inequalities hold for two dimensional space*

$$\|\nabla u\|_{L^2} \leq C\|u\|_{L^2}^{2/3} \|\nabla^3 u\|_{L^2}^{1/3}, \tag{8}$$

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2}, \tag{9}$$

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{5/6} \|\nabla^3 u\|_{L^2}^{1/6}. \tag{10}$$

Proof. See e. g., [8].

Lemma 2.5 *The following inequality holds:*

$$\|\nabla^m (u\cdot\nabla v) - u\cdot\nabla\nabla^m v\|_{L^2} \leq C\left(\|\nabla u\|_{L^\infty} \|\nabla^m v\|_{L^2} + \|\nabla v\|_{L^\infty} \|\nabla^m u\|_{L^2}\right). \tag{11}$$

Proof. See e. g., [8].

Lemma 2.6 *There exists a uniform positive constant C, such that*

$$\|\nabla u\|_{L^\infty} \leq C\left(1 + \|u\|_{L^2} + \|\nabla\times u\|_{BMO} \sqrt{\ln(e + \|u\|_{H^2})}\right) \tag{12}$$

holds for all vectors $u \in H^3(R^3)$ with $\nabla\cdot u = 0$.

Proof. More details of proof can see [8].

3 Results and Discussion

These are the main results of the paper.

Theorem 3.1 *Suppose that $(u_0, b_0) \in H^3(R^2)$ with $\nabla\cdot u_0 = 0,\ \nabla\cdot b_0 = 0$, (u, b) is a smooth solution to the Cauchy problem (1)-(4) for $0 \leq t < T$. Then (u, b) is smooth at time $t = T$ provided that the following condition holds*

$$\int_0^T \frac{\|\nabla\times u(t)\|_{BMO}}{\sqrt{\ln(e + \|\nabla\times u(t)\|_{BMO})}} dt < +\infty. \tag{13}$$

Theorem 3.2 *Suppose that $(u_0, b_0) \in H^m(R^2) (m \geq 3)$ with $\nabla\cdot u_0 = 0,\ \nabla\cdot b_0 = 0$, the result in Theorem 3.1 also holds.*

Proof of Theorem 3.1 Multiplying (1), (2) by $u,\ b$, using (3) and integrating in R^2, using integration by parts, we derive

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|_{L^2}^2+\|b\|_{L^2}^2\Big)+\nu\|\nabla u\|_{L^2}^2+\eta\|\nabla b\|_{L^2}^2=0. \tag{14}$$

Integrating with respect to t, we obtain

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+2\int_0^t\Big(\nu\|\nabla u(\tau)\|_{L^2}^2+\eta\|\nabla b(\tau)\|_{L^2}^2\Big)d\tau=\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2. \tag{15}$$

Applying ∇ to (1), (2), then taking inner product with $(\nabla u,\nabla v)$ in $L^2(R^2)$, using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\Big(\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\|\nabla b\|_{L^2}^2\Big)+\Big(\nu\|\nabla^2u(t)\|_{L^2}^2+\|\nabla^2b(t)\|_{L^2}^2\Big)\\&=-\int_{R^2}\nabla(u\cdot\nabla u)\nabla udx+\int_{R^2}\nabla(b\cdot\nabla b)\nabla udx\\&\quad-\int_{R^2}\nabla(u\cdot\nabla b)\nabla bdx+\int_{R^2}\nabla(b\cdot\nabla u)\nabla bdx.\end{aligned} \tag{16}$$

Using $\nabla\cdot u=0,\ \nabla\cdot b=0$, we derive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\Big)+\nu\|\nabla^2u(t)\|_{L^2}^2+\eta\|\nabla^2b(t)\|_{L^2}^2\\&\le C\|\nabla u\|_{L^\infty}\Big(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\Big).\end{aligned} \tag{17}$$

By Gronwall inequality and Lemma 2.6, we arrive

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+2\nu\int_{t_0}^t\|\nabla^2u(s)\|_{L^2}^2ds+2\eta\int_{t_0}^t\|\nabla^2b(s)\|_{L^2}^2ds\\&\le\Big(\|\nabla u(t_0)\|_{L^2}^2+\|\nabla b(t_0)\|_{L^2}\Big)e^{C\int_{t_0}^t\big(1+\|u(s)\|_{L^2}+\|\nabla\times u(s)\|_{BMO}\sqrt{\ln(e+\|u(s)\|_{H^3})}\big)ds}.\end{aligned} \tag{18}$$

From (13), there exist any small constant $\varepsilon>0$ and $T_*<T$ such that

$$\int_{T_*}^T\frac{\|\nabla\times u(t)\|_{BMO}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{BMO})}}dt\le\varepsilon. \tag{19}$$

Hence, from (17)-(19), we conclude that

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+2\nu\int_{t_0}^t\|\nabla^2u(s)\|_{L^2}^2ds+2\eta\int_{t_0}^t\|\nabla^2b(s)\|_{L^2}^2ds\\&\le C_0e^{C_1\int_{T_*}^t\|\nabla\times u(s)\|_{BMO}\sqrt{\ln(e+\|u(s)\|_{H^3})}ds}\\&\le C_0e^{C_1\varepsilon\ln(e+A(t))}\le C_2(e+A(t))^{C_1\varepsilon},\end{aligned} \tag{20}$$

where $A(t) = \sup_{T_* \le s \le t} (\|\nabla^3 u(s)\|_{L^2}^2 + \|\nabla^3 b(s)\|_{L^2}^2)$, $t \in [T_*, T]$, C_0 depends on $\|\nabla u(T_*)\|_{L^2}^2 + \|\nabla b(T_*)\|_{L^2}^2$, $C_1 > 0$ is an uniform constant.

Applying ∇^m to (1) and (2), then taking the L^2 inner product of the resulting equation with $\nabla^m u$ and $\nabla^m b$ respectively, then using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla^m u(t)\|_{L^2}^2 + \nu\|\nabla^m \nabla u(t)\|_{L^2}^2 = \int_{R^2} (\nabla^m (b\cdot\nabla b)\nabla^m u - \nabla^m (u\cdot\nabla u)\nabla^m u)dx, \tag{21}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla^m b(t)\|_{L^2}^2 + \eta\|\nabla^m \nabla b(t)\|_{L^2}^2 = \int_{R^2} (\nabla^m (b\cdot\nabla u)\nabla^m b - \nabla^m (u\cdot\nabla b)\nabla^m b)dx. \tag{22}$$

It follows from (21)-(22) and $\nabla\cdot u = \nabla\cdot b = 0$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^m u\|_{L^2}^2 + \|\nabla^m b\|_{L^2}^2\right) + \nu\|\nabla^m \nabla u(t)\|_{L^2}^2 + \eta\|\nabla^m \nabla b(t)\|_{L^2}^2 \\
&= \underline{-\int_{R^2}\left[\nabla^m (u\cdot\nabla u) - u\cdot\nabla\nabla^m u\right]\nabla^m u dx} + \underline{\int_{R^2}\left[\nabla^m (b\cdot\nabla b) - b\cdot\nabla\nabla^m b\right]\nabla^m u dx} \\
&\quad \underline{-\int_{R^2}\left[\nabla^m (u\cdot\nabla b) - u\cdot\nabla\nabla^m b\right]\nabla^m b dx} + \underline{\int_{R^2}\left[\nabla^m (b\cdot\nabla u) - b\cdot\nabla\nabla^m u\right]\nabla^m b dx} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{23}$$

Since the proof for the case $m > 3$ is similar to $m = 3$, here we only consider $m = 3$.

By the Hölder inequality, the Cauchy inequality and Lemma 2.4-2.5, we get

$$\begin{aligned}
I_1 &\le \|\nabla^3 (u\cdot\nabla u) - u\cdot\nabla^3 u\|_{L^2}\|\nabla^3 u\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^2}^2 \\
&\le \frac{\eta}{4}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e + A(t)),
\end{aligned} \tag{24}$$

$$I_2 \le \frac{\eta}{4}\|\nabla^4 b(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e + A(t)), \tag{25}$$

$$\begin{aligned}
I_3 &= -\int_{R^2}\left[\nabla^3 (u\cdot\nabla b) - u\cdot\nabla\nabla^3 b\right]\nabla^3 b dx \\
&= -3\int_{R^2}\nabla u\cdot\nabla\nabla^2 b\nabla^3 b dx - 3\int_{R^2}\nabla^2 u\cdot\nabla\nabla b\nabla^3 b dx - \int_{R^2}\nabla^3 u\cdot\nabla b\nabla^3 b dx \\
&\le 8\|\nabla u\|_{L^\infty}\|\nabla^3 b\|_{L^2}^2 + 8\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2}\|\nabla^4 b\|_{L^2} + \|\nabla^2 u\|_{L^4}\|\nabla b\|_{L^4}\|\nabla^4 b\|_{L^2} \\
&\le 8\|\nabla u\|_{L^\infty}\|\nabla^3 b\|_{L^2}^2 + C\Big(\|\nabla u\|_{L^\infty}\|\nabla^4 b\|_{L^2}^{4/3}\|\nabla b\|_{L^2}^{2/3}\Big) \\
&\qquad + C\Big(\|\nabla u\|_{L^\infty}^{1/2}\|\nabla^3 b\|_{L^2}^{1/2}\|\nabla^4 b\|_{L^2}^{5/6}\|\nabla b\|_{L^2}^{7/6}\Big) \\
&\le 8\|\nabla u\|_{L^\infty}\|\nabla^3 b\|_{L^2}^2 + \Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}^3\|\nabla b\|_{L^2}^2\Big) \\
&\qquad + \Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}^{6/5}\|\nabla b\|_{L^2}^2\|\nabla^3 u\|_{L^2}^{6/5}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq 8\|\nabla u\|_{L^\infty}\|\nabla^3 b\|_{L^2}^2 + \Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t))^{\frac{3}{2}C_2\varepsilon}A^{\frac{1}{2}}(t)\Big)\\
&\qquad+\Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t))^{\frac{21}{20}C_2\varepsilon}A^{\frac{13}{20}}(t)\Big)\\
&\leq 8\|\nabla u\|_{L^\infty}(e+A(t)) + \Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t))\Big)\\
&\qquad+\Big(\frac{\eta}{8}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t))\Big) \qquad (26)
\end{aligned}
$$

provide that $\varepsilon \leq \frac{1}{5C_2}$. Similarly, we have

$$
I_4 = -\int_{R^2}\Big[\nabla^3(b\cdot\nabla u) - b\cdot\nabla\nabla^3 u\Big]\nabla^3 b dx \leq \frac{\eta}{4}\|\nabla^4 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t)). \qquad (27)
$$

Thus, it follows from (23)-(27) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 b\|_{L^2}^2\Big) + \nu\|\nabla^4 u\|_{L^2}^2 + \eta\|\nabla^4 b\|_{L^2}^2\\
&\leq C\|\nabla u\|_{L^\infty}\left(e+A(t)\right) \qquad (28)
\end{aligned}
$$

for all $T_* \leq t < T$.

Integrating (28) over $[T_*, s]$ with respect to t, using Lemma 2.6, we conclude

$$
\begin{aligned}
&\|\nabla^3 u(s)\|_{L^2}^2 + \|\nabla^3 b(s)\|_{L^2}^2 + \nu\int_{T_*}^{s}\|\nabla^4 u(t)\|_{L^2}^2 dt + \eta\int_{T_*}^{s}\|\nabla^4 b(t)\|_{L^2}^2 dt\\
&\leq \|\nabla^3 u(T_*)\|_{L^2}^2 + \|\nabla^3 b(T_*)\|_{L^2}^2\\
&\quad+C\int_{T_*}^{s}\Big(1+\|u(t)\|_{L^2} + \|\nabla\times u(t)\|_{BMO}\sqrt{ln(e+\|u(t)\|_{H^3})}\Big)(e+A(t))dt\\
&\leq \|\nabla^3 u(T_*)\|_{L^2}^2 + \|\nabla^3 b(T_*)\|_{L^2}^2\\
&\quad+C\int_{T_*}^{s}\Big(1+\|u(t)\|_{L^2} + \|\nabla\times u(t)\|_{BMO}\sqrt{ln(e+A(t))}\Big)(e+A(t))dt, \qquad (29)
\end{aligned}
$$

which implies

$$
\begin{aligned}
e+A(t) &\leq e + \|\nabla^3 u(T_*)\|_{L^2}^2 + \|\nabla^3 b(T_*)\|_{L^2}^2\\
&\quad+C\int_{T_*}^{t}\Big(1+\|u(s)\|_{L^2} + \|\nabla\times u(s)\|_{BMO}\sqrt{ln(e+\|u(s)\|_{H^3})}\Big)(e+A(s))ds.
\end{aligned}
$$

For all $T_* \leq t < T$, using the Gronwall inequality and (29), we conclude that $e+A(t)$ is bounded, i.e.,

$$
\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 b(t)\|_{L^2}^2 \leq C, \qquad (30)
$$

where C is dependent on $\|\nabla^3 u(T_*)\|_{L^2}^2 + \|\nabla^3 b(T_*)\|_{L^2}^2$.

Thus, we have completed the proof of Theorem 3.1.

The proof of Theorem 3.1 is similar to the proof of Theorem 3.1, we only need to set $m = 3$ in (23).

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