Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 2, 85 - 91
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A Common Fixed Point Theorem in Fuzzy 2- Metric Space

Surjeet Singh Chauhan[1] (Gonder) and Kiran Utreja[2]

[1] Dept. of Applied Science and Humanities
Chandigarh University, Gharuan, India
surjeetschauhan@yahoo.com

[2] Dept. of Applied Science and Humanities, GNIT, Mullana, India
kiranutreja41@gmail.com

Abstract

In this paper we prove a common fixed point theorem in fuzzy 2- metric space on six self-mappings using the concept of sub compatibility of type A and commutativity changing the concept of sub-compatibility introduced by Kamal Wadhwa et al. [6].

Mathematics Subject Classification: 47H10, 54H25, 54A40

Keywords: Fixed point, fuzzy-2 metric space

1. Introduction

In 1965, L.A. Zadeh [7] introduced the concept of fuzzy sets which became active field of research for many researchers. In 1975, Kramosil and Michalek [5] came in front with the concept of Fuzzy metric space based on fuzzy sets which were further modified by George and Veermani [2] with the help of t-norms. Many authors did good work and are still doing in proving fixed point theorems in Fuzzy metric space. Singh and Chauhan [3] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces. Jungck et al. [4] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems.

Using the concept of compatible maps of type (A), Jain et al. [1] proved a fixed point theorem for six self- maps in a fuzzy metric space, generalizing the result of Cho [8].Kamal Wadhwa et al. [6] introduce the new concepts of sub-compatibility and sub sequential continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity in Fuzzy metric space using Implicit Relation and establish a common fixed point theorem. In this we are introducing the new version of sub-compatibility with the name sub-compatible mapping of type A to prove the result and proving are fact by siting an example on six self-mappings

2. Preliminaries

Def.2.1. A binary operation *: [0, 1] × [0, 1] → $[0,1]$ is called a t-norm in ([0, 1], *) if for all a, b, c, d∈ $[0,1]$following conditions are satisfied:

i. a*1 = a,
ii. a*b = b*a,
iii. a*b≤c*d whenever a≤c and b≤ d,
iv. a*(b*c) = (a*b)*c.

Def.2.2. The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t- norm and M is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions for all x, y, z∈X and s, t > 0

i. M(x, y, 0) = 0,
ii. M(x, y, t) = 1 ∀t>0 iff x= y,
iii. M(x, y, t) = M(y, x, t),
iv. M(x, y, t) * M(y, z, s) ≤ M (x, z, t+s)
v. M(x, y, .):[0, ∞)→[0, 1] is left continuous ,
vi. $\lim_{t\to\infty} M(x,y,t)$= 1.

E.g. Let (X, d) be a metric space. Define a*b = ab or a*b = min {a, b} and ∀x, y ∈ X, M (x, y, t) =$\frac{t}{t+d(x,y)}$.
Then (X, M, *) is a fuzzy metric space and the metric d is the standard fuzzy metric.

Def.2.3. A binary operation *: [0, 1] × [0, 1] × [0, 1] → [0,1] is called a continuous t- norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a_1*b_1*c_1 \leq a_2*b_2*c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 in [0,1].

Def.2.4. The 3-tuple (X, M, *) is called a fuzzy 2- metric space if X is an arbitrary set, * is a continuous t- norm and M is a fuzzy set in $X^3 \times [0,\infty]$ satisfying the following conditions $\forall$ x, y, z, u $\in$X and $t_1, t_2, t_3 > 0$

i. M(x, y, z,0) = 0,
ii. M(x, y, z, t) = 1, t > 0 and when at least two of the three points are equal,
iii. M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) (symmetry about three variables),
iv. $M(x, y, z, t_1+t_2+t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$ (This corresponds to tetrahedron inequality in 2-metric space)
v. M(x, y, z, .):$[0, \infty) \to [0, 1]$ is left continuous.

Def.2.5. A sequence $\{x_n\}$ in a fuzzy 2- metric space (X, M, *) is said to converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ $\forall$ a $\in$ X and t > 0.

Def.2.6. Let (X, M, *) be a fuzzy 2- metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ $\forall$ a $\in$ X, p > 0 and t > 0.

Def.2.7. A fuzzy 2- metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Def.2.8. Self – mappings S and T of a fuzzy 2- metric space (X, M, *) are said to weakly commuting if
M (STx, TSx, z, t) $\geq$ M (Sx, Tx, z, t) $\forall$ x$\in$ X and t > 0.

Def.2.9. Self- mappings S and T of a fuzzy 2- metric space (X, M, *) are said to be compatible if and only if M (STx$_n$, TSx$_n$, z, t) $\to$ 1 $\forall$ t > 0 whenever $\{x_n\}$ is a sequence in X such that $Tx_n, Sx_n \to p$ for some p in X as $n \to \infty$.

Def. 2.10. Suppose S and T be self –mappings of a fuzzy 2- metric space (X, M, *). A point x in X is called a coincidence point of S and T iff Sx = Tx, then w = Sx = Tx is called a point of coincidence of Sand T.

Def. 2.11. Self- maps S and T of a fuzzy 2- metric space (X, M, *) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points that is if Sp = Tp for some p $\in$ X then STp = TSp.

Remarks: Two compatible maps are weakly compatible but the converse is not true.

Def.2.12. Self -maps S and T of a fuzzy 2- metric space (X, M, *) are said to be occasionally weakly compatible if and only if there is a point x in X which is coincidence point of S and T at which they commute.

Def.2.13. Self-mappings S and T of a fuzzy 2- metric space (X, M, *) are said to sub compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ = z, z ∈ X and satisfy $\lim_{n\to\infty} M(STx_n, TS\,x_n, a, t) = 1$.

Def.2.14. Self-mappingsS and T of a Fuzzy 2-metric space are said to be sub-compatible of type A if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ = z, z ∈X and satisfy $\lim_{n\to\infty} M(STx_n, TT\,x_n, a, t) = 1$ and $\lim_{n\to\infty} M(TSx_n, SS\,x_n, a, t) = 1$.

3. Main Results

Lemma3.1. M(x, y, z, .) is non-decreasing for all x, y, z ∈ X.

Lemma3.2. Let (X, M, *) be a Fuzzy 2- metric space. If there exists k∈ (0, 1) such that M(x, y, z, kt) ≥ M(x, y, z, t) for all x, y, z ∈X with z≠ x, z≠ y and t >0, then x = y.

Theorem 3.3: Let A, B, P, Q, S and T be six self-maps of a Fuzzy 2-metric space (X, M, *) with continuous t-norm defined by t*t ≥ t for all t ∈ [0, 1]. If the pairs (AB, S) and (PQ, T) are sub compatible of type A having the same coincidence point and

AB = BA, BS=SB, AS =SA, PQ =QP, TQ =QT, PT =TP

then for all x, y, z ∈X, k ∈ (0, 1), t > 0

M(Sx, Ty, z, kt) ≥ { M(Sx, PQy, z,t)*M(Sx, ABx, z, t)*M(PQy, Ty, z,t) *M(ABx, PQy, z, t)*M(ABx, Ty, z, t)}.

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof: Since the pairs (AB, S) and (PQ, T) are sub compatible of type A then there exist two sequences $\{x_n\}$, $\{y_n\}$in X such that

$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Sx_n$= a, a∈ X and satisfy

$\lim_{n\to\infty} M(ABSx_n, SSx_n, z, t) = 1$ and $\lim_{n\to\infty} M(SABx_n, ABABx_n, z, t) = 1$.

Thus we have $\lim_{n\to\infty} M(ABa, Sa, z, t) = 1$ and $\lim_{n\to\infty} M(Sa, ABa, z, t) = 1$.

And $\lim_{n\to\infty} PQy_n = \lim_{n\to\infty} Ty_n$= b, b∈X and satisfy

$\lim_{n\to\infty} M(PQTy_n, TTy_n, z, t) = 1$ and $\lim_{n\to\infty} M(TPQy_n, PQPQy_n, z, t) = 1$.

Thus we have $\lim_{n\to\infty} M(PQb, Tb, z, t) = 1$ and $\lim_{n\to\infty} M(Tb, PQb, z, t) = 1$.

Therefore ABa = Sa and PQb = Tb.

Thus we have 'a' is coincidence point of AB and S, 'b' is coincidence point of PQ and T.
Now we are to prove a = b, for this take $x = x_n$ and $y = y_n$,

$M(Sx_n, Ty_n, z, kt) \geq \{M(Sx_n, PQy_n, z, t)*M(Sx_n, ABx_n z, t)*M(PQy_n, Ty_n, z, t)$
$*M(ABx_n, PQy_n, z, t)*M(ABx_n, Ty_n, z, t)\}$.

Take the limit as $n\to \infty$, we get

$M(a, b, z, kt) \geq \{M(a, b, z, t)*M(a, a, z, t)*M(b, b, z, t)*M(a, b, z, t)$
$*M(a, b, z, t)\}$

This implies $M(a, b, z, kt) \geq M(a, b, z, t)$ for all $t > 0$.
Thus by lemma 3.2 a = b which shows that AB, S, PQ, T have the same coincidence point.
Next we are to prove Aa = Ba = Pa = Qa = Sa = Ta = a.
First take x = a and $y = y_n$

$M(Sa, Ty_n, z, kt) \geq \{M(Sa, PQy_n, z, t)*M(Sa, ABa, z, t)*M(PQy_n, Ty_n, z, t)$
$*M(ABa, PQy_n, z, t)*M(ABa, Ty_n, z, t)\}$.

Take the limit as $n\to \infty$, we get

$M(Sa, b, z, kt) \geq \{M(Sa, b, z, t)*M(Sa, a, z, t)*M(b, b, z, t)*M(a, b, z, t)$
$*M(a, b, z, t)\}$.

As a = b, we get
$M(Sa, a, z, kt)\geq M(Sa, a, z, t)$ which gives Sa = a.
Now take $x = x_n$ and y = a,

$M(Sx_n, Ta, z, kt) \geq \{M(Sx_n, PQa, z, t)*M(Sx_n, ABx_n, z, t)*M(PQa, Ta, z, t)$
$*M(ABx_n, PQa, z, t)*M(ABx_n, Ta, z, t)\}$.

Take the limit as $n\to \infty$, we get

$M(a, Ta, z, kt) \geq \{M(a, Ta, z, t)*M(a, a, z, t)*M(Ta, Ta, z, t)*M(a, Ta, z, t)$
$*M(a, Ta, z, t)\}$.

We get
$M(a, Ta, z, kt) \geq M(a, Ta, z, t)$ which gives Ta = a.
Next we are to prove Aa = Ba= a.
Put x = Ba and $y = y_n$

$M(SBa, Ty_n, z, kt) \geq \{M(SBa, PQy_n, z,t)*M(SBa, ABBa, z, t)$
$*M(PQy_n, Ty_n, z,t)*M(ABBa, PQy_n, z, t)*M(ABBa, Ty_n, z, t)\}$.

As A, B and S commute thus ABBa = BABa =BSa =Ba and SBa= BSa =Ba.

$M(Ba, a, z, kt) \geq \{M(Ba, a, z,t)*M(Ba, Ba, z, t)*M(a, a, z,t)$
$*M(Ba, a, z, t)*M(Ba, a, z, t)\}$.

$M(Ba, a, z, kt) \geq M(Ba, a, z, t)$.
Therefore Ba = a.
Now Put x = Aa and $y = y_n$

$M(SAa, Ty_n, z, kt) \geq \{M(SAa, PQy_n, z,t)*M(SAa, ABAa, z, t)$
$*M(PQy_n, Ty_n, z,t)*M(ABAa, PQy_n, z, t)*M(ABAa, Ty_n, z, t)\}$.

As A, B and S commute thus ABAa = ASa = Aa and SAa= ASa =Aa.

$M(Aa, a, z, kt) \geq\{M(Aa, a, z,t)*M(Aa, Aa, z, t)*M(a, a, z,t)$

*M(Aa, a, z, t)*M(Aa, a, z, t)}.
M (Aa, a, z, kt)≥ M (Aa, a, z, t).
Therefore Aa= a.
Thus Aa = Ba = Sa = a.
Left is to prove Pa = Qa = Ta = a.
For this take x = x_n and y = Qa to prove Qa = a and to prove Pa = a take x = x_n and y = Pa.
Thus Aa = Ba = Pa= Qa= Sa= Ta = a.

Example: Let A, B, S, T, P, Q be self-mappings of X = [0, 1] where
$Ax=\frac{x}{5}$ $Bx=\frac{x}{3}$ $Sx=\frac{x}{15}$
$Tx=\frac{x}{15}$ $Px=\frac{x}{6}$ $Qx=\frac{2x}{5}$
Let $\{x_n\}$ and $\{y_n\}$ be two sequences where
$x_n=\frac{n}{n+1}$ and $y_n=\frac{n}{n-1}$
Then 1/15 is fixed point of A, B, S, T, P and Q.

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Received: October, 2012