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Maximum and Minimum Fixed Points of Isotone Operators in Partially Ordered Sets

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Abstract. We establish common fixed point theorems for a family of isotone selfmaps of a poset by generalizing known previous results. The independence of two theorems is also shown under suitable examples.

Keywords: common fixed point; isotone map; Zorn lemma

1. Preliminaries and Main results

Let $P = (P,\leq)$ be a poset and F be a family of selfmaps of P. In what follows we shall use also the set of the common fixed points of the family F, that is the set $\Phi_P(F) = \{x\in P : f(x)=x$ for every $f\in F\}$. A selfmap of P is said isotone if $a \leq b$ implies $f(a) \leq f(b)$ with $a,b\in P$. Following the main ideas of [2, 3, 4], we establish two important results:

Theorem A. Let P be a a poset and F be a family of isotone selfmaps of P satisfying the following properties:
(A1) there exists some element $x_0\in P$ such that $x_0 \leq f(x_0)$ for any $f\in F$,
(A2) if C is a chain of P not having supremum in P, then there exists $\sup\{h_x(x): x\in C$ for some $h_x\in F\} = \sigma$ for some $\sigma\in P$,
(A3) if $x\leq h_x(x)$ for some $h_x\in F$, then $h_x(x) \leq f(h_x(x))$ for any $f\in F$.
Then there exists an element $m_0\in P$ such that $m_0= \min\{m: m\in\Phi_P(F)\cap P^+(x_0)\}$, where $P^+(x_0)=\{x\in P: x_0 \leq x\}$.

Proof. Define the set $P(F) = \{x\in P: x \leq f(x)$ for any $f\in F\}$. By property (A1), we have that $x_0\in P(F)$. Hence $P^+(x_0)\cap P(F) \neq\emptyset$ and let Γ be a chain of this set not having supremum in P. By property (A2), let $\sup\{h_c(c): c\in \Gamma$ for some $h_c\in F \}= \sigma$ for some $\sigma\in P$. We have that $x_0 \leq c \leq h_c(c) \leq \sigma$ for any $c\in\Gamma$. Being F a family of isotone selfmaps of P, it follows that $h_c(c) \leq f(h_c(c)) \leq f(\sigma)$ for any $c\in\Gamma$ and for any $f\in F$ by property (A3). This means that $\sigma \leq f(\sigma)$ for every $f\in F$ and hence $\sigma\in P^+(x_0)\cap P(F)$ is an upper bound of Γ. Then every chain Γ of $P^+(x_0) \cap$

P(F) has an upper bound and there exists a maximal element m such that m = f(m) for every f∈F by Zorn's lemma, that is $m\in \Phi_P(F)\cap P^+(x_0)$. An iterated application of Zorn's lemma to the set $M_P(F,x_0) = \{y\in P(F): x_0 \le y \le m$ for every $m\in\Phi_P(F)\cap P^+(x_0)\}$ ($M_P(F,x_0) \neq\emptyset$ because $x_0\in P(F)$) gives the thesis.

Example 1. Property (A3) is necessary in Theorem A otherwise the thesis is false. Indeed let P=[0,1] with the natural order, F={f,g} be defined from $f(x)= k_1$, $g(x)= k_2$ for any $x\in P$, $0\le k_1< k_2\le 1$. Then all the assumptions of Theorem A are trivially satisfied except property (3) because if x is such that $0\le k_1<x\le k_2$, we have that for $g=h_x$, it follows that $k_1<x\le k_2= h_x(x) \le f(h_x(x)) = k_1$, a contradiction (cfr., [3]).

We illustrate Theorem A with the following example (cfr., [3]):

Example 2. Let P = [0,1]-{ ½} and F={f,g} where f,g: P→P are defined as f(x) = x/2, g(x) = 1/4 if $0 \le x < ½$ and f(x)= (x+1)/2, g(x)=x if $½< x \le 1$. Note that F={f,g} is a family of isotone selfmaps of P and the unique chains of P not having supremum are those ones just having $½\notin P$ as a supremum in R (reals). If Γ is such a chain, then we have $\sup\{h_c(c): c\in\Gamma$ for some $h_c\in F\}=¼$ if we take, for example, $h_c=g$ for any c∈C, thus property (A2) holds. We observe that an element x_0 (resp., x) satisfies properties (A1) (resp., (A3)) if and only if x_0 resp.,x) belongs to (½,1], further $\Phi_P(F) = \{1\}$. Then all the assumptions (A1), (A2), (A3) are satisfied and indeed $m_0= 1$ since for any $x_0\in(½,1]$.

Theorem B. Let P be a poset and F be a family of isotone selfmaps of P satisfying the following properties:
(B1) there exists some element $x_0\in P$ such that $x_0 \le f(x_0)$ and $f(g(x_0))= g(f(x_0))$ for any f,g∈F,
(B2) if C is a chain of P not having supremum in P, then there exists $\sup\{h_x(x): x\in C$ for some $h_x\in F\} = \sigma$ for some $\sigma\in P$ such that $f(g(\sigma))= g(f(\sigma))$ for any f,g∈F.
Then there exists an element $m_0\in P$ such that $m_0= \min\{m: m\in\Phi_P(F)\cap P^+(x_0)\}$.

Proof. Define the set $Q(F) = \{x\in P: x \le f(x)$ and $f(g(x))= g(f(x))$ for any $f,g\in F\}$. By property (A1), we have that $x_0\in Q(F)$. Hence $P^+(x_0)\cap Q(F) \neq\emptyset$ and let Γ be a chain of this set not having supremum in P. By property (B1), let $\sup\{h_c(c): c\in\Gamma$ for some $h_c\in F\} = \sigma$ for some σ∈P such that $f(g(\sigma))= g(f(\sigma))$ for any f,g∈F. We have that $x_0 \le c \le h_c(c) \le \sigma$ for any c∈Γ. Being F a family of isotone selfmaps of P from $c \le f(c)$ for any c∈Γ, it follows that $h_c(c) \le h_c(f(c)) = f(h_c(c)) \le f(\sigma)$ for any c∈Γ and for any f∈F by property (B2). This means that $\sigma \le f(\sigma)$ for every f∈F and hence $\sigma\in P^+(x_0)\cap Q(F)$ is an upper bound of Γ. Then every chain Γ of $P^+(x_0) \cap Q(F)$ has an upper bound and there exists a maximal element m such that m = f(m) for every f∈F by Zorn's lemma, that is $m\in \Phi_P(F)\cap P^+(x_0)$. As above in the proof of Theorem A, an iterated application of Zorn's lemma to the set $M_Q(F,x_0) = \{y\in Q(F): x_0 \le y \le m$ for every $m\in\Phi_P(F)\cap P^+(x_0)\}$ ($M_Q(F,x_0) \neq\emptyset$ because $x_0\in Q(F)$) gives the thesis.

Corollary C. Let P be a a poset, F be a family of isotone selfmaps of P satisfying the following properties:

(C1) ≡ (B1),
(C2) if C is a chain of P not having supremum in P, then there exist some $h \in F$, $\sigma \in P$ such that $\sup h(C) = \sigma$ and $f(g(\sigma))= g(f(\sigma))$ for any $f,g \in F$.
Then there exists an element $m_0 \in P$ such that $m_0= \min\{m: m \in \Phi_P(F) \cap P^+(x_0)\}$.

Proof. It suffices to take $h_c = h$ for any $c \in C$ and apply Theorem B.

Remark D. Corollary C generalizes Theorem A of [7], where the assumption "$f(g(\sigma))= g(f(\sigma))$ for any $f,g \in F$" was missing and this corrects that mere error.

We illustrate Theorem B (resp., Corollary C) with the following example (cfr., [3]):

Example 3. Let $P = [0,1]-\{ ½\}$ and $F = \{f,g\}$ where $f,g:P \to P$ are defined as $f(x) = x$, $g(x) = ¼$ if $0 \leq x < ½$ and $f(x)= (x+1)/2$, $g(x)=x$ if $½ < x \leq 1$. Note that $F=\{f,g\}$ is a family of commutative isotone selfmaps of P and the unique chains of P not having supremum are those ones just having $½ \notin P$ as a supremum in R (reals). If Γ is such a chain, then we have $\sup \{h_c(c): c \in C \text{ for some } h_c \in F\}=¼$ if we take, for instance, $h_c=g$ for any $c \in \Gamma$, thus property (B2) (resp., (C2)) holds. We observe that an element x_0 satisfies property (B1) (resp., (C1)) if and only if $x_0 \in [0, ¼] \cup (½,1]$, moreover $\Phi_P(F) = \{¼, 1\}$. Then all the assumptions of Theorem B (resp., Corollary C) are satisfied: indeed, $m_0= ¼$ if $x_0 \in [0, ¼]$ and $m_0= 1$ if $x_0 \in (½,1]$.

Remark E. Note that Theorem A is not applicable to Example 3 because property (A3) does not hold. Indeed, if $x \in (¼,½)$, we have $x=h_x(x)$ if we take $h_x = f$ and we should have $¼ < x= h_x(x) \leq g(h_x(x)) = ¼$, a contradiction. We also note that Theorem B is not applicable to Example 2 because property (B2) does not hold. Indeed, if C is a chain such that $\sup C = ½ \notin P$ but $\sup\{h_c(c): c \in C \text{ for some } h_c \in F\}=¼$ if we take, for instance, $h_c=g$ for any $c \in C$, we should have $⅛ = f(g(¼)) = g(f(¼)) = ¼$, a contradiction. Therefore Theorem A and Theorem B are two results independent from each other.

2. Dual results

Dually we can prove the following results:

Theorem F. Let P be a a poset and F be a family of isotone selfmaps of P satisfying the following properties:
(F1) there exists some element $x_0 \in P$ such that $x_0 \geq f(x_0)$ for any $f \in F$,
(F2) if C is a chain of P not having infimum in P, then there exists $\inf\{h_x(x): x \in C \text{ for some } h_x \in F\} = \lambda$ for some $\lambda \in P$,
(A3) if $x \geq h_x(x)$ for some $h_x \in F$, then $h_x(x) \geq f(h_x(x))$ for any $f \in F$.
Then there exists an element $M_0 \in P$ such that $M_0= \max\{M: M \in \Phi_P(F) \cap P^-(x_0)\}$, where $P^-(x_0)=\{z \in P: x_0 \geq z\}$.

Theorem G. Let P be a a poset and F be a family of isotone selfmaps of P satisfying the following properties:
(G1) there exists some element $x_0 \in P$ such that $x_0 \geq f(x_0)$ and $f(g(x_0))= g(f(x_0))$ for any $f,g \in F$,
(G2) if C is a chain of P not having infimum in P, then there exists $\inf\{h_x(x): x \in C$ for some $h_x \in F\} = \lambda$ for some $\lambda \in P$ such that $f(g(\lambda))= g(f(\lambda))$ for any $f,g \in F$.
Then there exists an element $M_0 \in P$ such that $M_0= \max\{M: M \in \Phi_P(F) \cap P^-(x_0)\}$.

Corollary H (to Theorem G). Let P be a poset and F be a family of isotone selfmaps of P satisfying the following properties:
(H1) ≡ (G1),
(H2) if C is a chain of P not having infimum in P, then there exist some $h \in F$, $\lambda \in P$ such that $\inf h(C) = \lambda$ and $f(g(\lambda))= g(f(\lambda))$ for any $f,g \in F$.
Then there exists an element $M_0 \in P$ such that $M_0= \max\{M: M \in \Phi_P(F) \cap P^-(x_0)\}$.

We illustrate Theorem G or Corollary H with the following example:

Example 4. Let P={a,b,c,d,e} be a set in which the following partial ordering is defined:

$a \leq b$, $a \leq c$, $a \leq d$, $a \leq e$, $b \leq c$, $b \leq e$, $d \leq e$.

Then $(P,\leq)$ is a finite poset with minimum equal to a. Let F={f} where $f: P \rightarrow P$ is defined as $f(a) = a$, $f(b) = f(d) = b$, $f(c) = a$, $f(e) = e$. It is easily verified that f is isotone with respect to the given partial ordering. The property (G2) of Theorem G (resp., property (H2) of Corollary H) is satisfied vacuosly since any chain of P is finite and hence has a minimum. Then all the assumptions of the Theorem G (resp., Corollary H) hold being F singleton and we find that $\Phi_P(f) = \{a,b,e\}$. Then $M_0 = a$ if $x_0 = a$, $M_0 = b$ if $x_0 = b$, $M_0 = e$ if $x_0 = e$ and $M_0 = b$ if $x_0 = c$. We point out that P is not a complete lattice with respect to the given partial ordering because the subset {c,e} has no supremum.

Remark I. Corollary H generalizes Theorem B of [7], where the assumption "$f(g(\lambda))= g(f(\lambda))$ for any $f,g \in F$" was missing and this corrects that mere error.

We also have this simple and useful result.

Theorem L. Let P be a poset with minimum and maximum and F be a family of isotone selfmaps of P satisfying the following properties:

(L2') if C is a chain of P not having supremum in P, then there exist some $h \in F$, $\sigma \in P$ such that $\sup h(C) = \sigma$ and $f(g(\sigma))= g(f(\sigma))$ for any $f,g \in F$.
(L2'') if Γ is a chain of P not having infimum in P, then there exist some $m \in F$, $\lambda \in P$ such that $\inf m(\Gamma) = \lambda$ and $f(g(\lambda))= g(f(\lambda))$ for any $f,g \in F$.

Then there exist $m_0 = \min \Phi_P(F)$ and $M_0 = \max \Phi_P(F)$.

Proof. An application of Corollary C (resp., H) to the set $\Phi_P(F)\cap P^+(m)$ (resp., $\Phi_P(F)\cap P^-(M)$) gives the thesis.

Example 5. Let P={a,b,c,d,e} be a set in which the following partial ordering is defined:

$a \leq b$, $a \leq c$, $a \leq d$, $a \leq e$, $b \leq c$, $b \leq e$, $c \leq e$, $d \leq e$.

Then $(P,\leq)$ is a finite poset with minimum equal to a and maximum e. Let F={f} where f: P→ P is defined as f(a) = a, f(b) = f(d) = b, f(c) = a, f(e) = e. It is easily verified that f is isotone with respect to the given partial ordering. The properties (L2') and (L2'') hold trivially since any chain C ($\neq$ P) of P is finite and hence C has a minimum and a maximum. Then all the assumptions of the Theorem I hold and we find that $\Phi_P(f)$ = {a,b,e}, therefore m_0 = a and M_0 = e. Note that the itself P is not a chain because c and d are not comparable.

We terminate with another example which illustrates Theorem L.

Example 6. Let P and F be as in Example 3. P is a poset with minimum 0 and maximum 1. The unique chains of P not having supremum (resp., infimum) are those ones $C=\{x \in P: x < ½\}$(resp., $\Gamma=\{x \in P: x > ½\}$) such that sup C = ½$\notin$ P (resp., inf Γ = ½$\notin$ P) in R (reals). Then we have sup h(C) =¼ (resp., inf m(Γ) = 3/4) if we take h = g (resp., m = f), thus property (L2') (resp., (L2'')) holds. Thus $\Phi_P(F)$ = {¼, 1}, that is m_0= ¼ and M_0= 1.

Remark M. It is possible to extend the actual results to the case of isotone multifunctions defined on posets, see, for instance, [5, 6] and to antitone operators [1, 3].

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