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Fourier Series Development for Solving a Steady Fluid Structure Interaction Problem

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Abstract

This paper presents a new approximation of pressures on the interface between fluid and structure based on Fourier series development. This approximation gives easily an analytic solution to structure equation. Using the least squares method, we introduce an optimization problem to determine Fourier coefficients series and deduct the structure displacement, the fluid velocity and the fluid pressure. This present work aims to extend and improve the approximation method of pressures on the interface.

Mathematics Subject Classification: 74F10, 65T40, 37M05

Keywords: Fluid-Structure, Fourier series development, Optimization

1 Introduction

Problems involving fluid structure interaction occur in a wide variety of engineering problems and therefore have attracted the interest of many investigations from different engineering disciplines. As a results, much efforts has gone into the development of general computational method for fluid structure systems [15], [16], [2], [5], [6],[13], [3], [9], [4], [10], [11], [12].

We saw in a previous paper [13] that pressures on the interface was approximated by a linear combination of shape functions as well as by a development of Fourier sine series in [11]. In [13], fluid-structure interaction problem has been solved in the reference domain Ω_0^F.

Thus, this paper aims at showing a new approximation method of pressures on the interface between fluid and structure based on Fourier cosine series development. This appproximation gives easily an analytic solution to structure equation. The latter depends on Fourier coefficients series. Using the least squares method, we establish an optimization problem to compute Fourier coefficients and obtain the structure displacement, the fluid velocity and the fluid pressure. In this work, we note that the fluid structure interaction problem is solved in the moving domain Ω_u^F. To solve this optimization problem by the quasi-Newton BFGS method we need the derivative of the cost function. Despite, it is a difficult task, the gradient of the cost function is computed in this work. In addition, the fluid is modeled by two dimensional Stoke equations for steady flow and the structure is represented by the one dimensional beam equation.

2 Fluid structure interaction problem

We denote by Ω_u^F the two-dimensional domain occupied by the fluid, and Γ_u the elastic interface between fluid and structure and $\Gamma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ be the remaining external boundaries of the fluid as depicted Fig. 1.

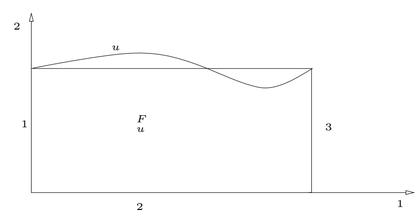

Figure 1: Sets appearing to the fluid structure interaction problem

Moreover, the reference domain is defined by $\Omega_0^F = [0, L] \times [0, H]$.
The coupled problem is to find (u, v, p) such that:

$$EI\frac{d^4u}{dx_1^4} = \left(-(\sigma(p,v)\cdot n)\cdot \vec{e_2}\sqrt{1+(u\prime(x_1))^2}\right)\ for\, 0 < x_1 < L \tag{1}$$

$$u(0) = u(L) = \frac{du(0)}{dx_1} = \frac{du(L)}{dx_1} = 0 \tag{2}$$

$$-\mu\Delta v + \nabla p = f^F, \quad in \quad \Omega_u^F \tag{3}$$

$$\nabla \cdot v = 0, \quad in \quad \Omega_u^F \tag{4}$$

$$v = g, \quad on \quad \Sigma_1 \tag{5}$$

$$-pI_2 n + \mu\nabla v \cdot n = 0, \quad on \quad \Sigma_3 \tag{6}$$

$$v = 0, \quad on \quad \Gamma_u \tag{7}$$

$$v_2 = 0, \quad on \quad \Sigma_2 \tag{8}$$

$$\frac{\partial v_1}{\partial x_2} = 0, \quad on \quad \Sigma_2 \tag{9}$$

Where,

- E is the young modulus,
- I is the moment of inertia,
- I_2 is the identity matrix,
- μ is the fluid viscosity,
- v_1 first component of v,
- v_2 second component of v,
- the vector $\vec{e_2} = (0, 1)$,
- n is the unit outward normal vector,
- Σ_2 is the symetric axis,
- On Σ_2, we have the non penetration condition: $v \cdot n = v_2 = 0$,
- On Σ_2, we have the continuity of Cauchy shear stress: $\sigma \cdot n = \frac{\partial v_1}{\partial x_2} = 0$,
- g the velocity profil in inflow Σ_1.

We assume that $(-(\sigma(p, v) \cdot n) \cdot \vec{e_2}\sqrt{1 + (u\prime(x_1))^2} = p(x_1, H + u(x_1))$ [13], [9]. As a consequence, equation (1) becomes $EI\frac{d^4u}{dx_1^4} = p(x_1, H + u(x_1))$.

3 Fourier cosine series development

We assume that on the interface between the fluid and the structure:

$$p(x_1, H + u(x_1)) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega x_1). \tag{10}$$

Where

- $\omega = \frac{2\pi}{T}$ is the frequency,
- $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ is Fourier cosine coefficients series.

Now, to have numerical results, the partial sum is used:

$$p(x_1, H + u(x_1)) \approx \alpha_0 + \sum_{n=1}^{N} \alpha_n \cos(n\omega x_1). \tag{11}$$

Where $N \gg 1$. Thus, this approximation gives the structure displacement u as follows:

$$u(x_1) = \frac{1}{EI}\left(\alpha_0 \frac{x_1^4}{24} + \sum_{n=1}^{N} \frac{\alpha_n}{(n\omega)^4}\cos(n\omega x_1) + k_1\frac{x_1^3}{6} + k_2\frac{x_1^2}{2} + k_3 x_1 + k_4\right), \tag{12}$$

Thanks to equation (2) we determine constants k_1, k_2, k_3, k_4. Using the least squares method, we introduce the cost functional J to find Fourier coefficients. We have the following optimization problem:

$$\inf J(\alpha) = \int_0^L \left(\sum_{n=0}^{N} \alpha_n \cos(n\omega x_1) - p(x_1, H + u(x_1))\right)^2 dx_1, \tag{13}$$

subject to equations (12)-(2) and (3)-(9).

To solve this optimization problem by the quasi-Newton BFGS method we need the derivative of J with respect to α_k, $k \in \{0, 1, \ldots, N\}$.

4 Computation of gradient

Let α_k the k-th component of the vector α, we compute for each k in $\{0, 1, \ldots, N\}$ the partial derivative of J with respect to α_k as follow:

$$\frac{\partial J}{\partial \alpha_k} = 2\int_0^L \left(\sum_{n=0}^{N} a_n \cos(n\omega x_1) - p(x_1, H + u(x_1))\right)\left(\cos(k\omega x_1) - \frac{\partial p(x_1, H + u(x_1))}{\partial \alpha_k}\right) dx_1$$

To compute the term $\frac{\partial p(x_1, H+u(x_1))}{\partial \alpha_k}$, we introduce two propositions [13], [9].

Proposition 1 *Find $v \in (H^1(\Omega_u^F))^2$, $v = \overline{g}$ on $\partial\Omega_u^F$ and $p \in L^2(\Omega_u^F)/\mathbf{R}$ such that:*

$$\left\{\begin{array}{rcl} \int_{\Omega_u^F} \mu \nabla v \cdot \nabla w dx - \int_{\Omega_u^F} (\nabla \cdot v) p\, dx &=& \left\langle f^F, w \right\rangle, \quad \forall\, w \in (H_0^1(\Omega_u^F))^2 \\ - \int_{\Omega_u^F} (\nabla \cdot v) q\, dx &=& 0, \quad \forall\, q \in L^2(\Omega_u^F)/\mathbf{R} \end{array}\right. \tag{14}$$

have a unique solution.

Thanks to the transformation $T_u : \overline{\Omega}_0^F \to \overline{\Omega}_u^F$ such that:

$$T_u(\widehat{x}_1, \widehat{x}_2) = (x_1, x_2) = \left\{\begin{array}{ll} x_1 = \widehat{x}_1, & \forall (\widehat{x}_1, \widehat{x}_2) \in \Omega_0^F \\ & \\ x_2 = \frac{H + u(\widehat{x}_1)}{H}\widehat{x}_2, & \forall (\widehat{x}_1, \widehat{x}_2) \in \Omega_0^F \end{array}\right. \tag{15}$$

we write the proposition 1 in the reference domain Ω_0^F in order to partial derivatives under integral [13], [9]. After derivation, we use again the transformation T_u to write equations in the moving domain Ω_u^F. Then, we deduct the following proposition.

Proposition 2 *Applications $\alpha \in \mathbf{R}^m \mapsto v \in (H^1(\Omega_u^F))^2$ and $\alpha \in \mathbf{R}^m \mapsto p \in L^2(\Omega_u^F)/\mathbf{R}$ are differentiable and there partial derivative $\frac{\partial v}{\partial \alpha_k} \in (H^1(\Omega_u^F))^2$ and $\frac{\partial p}{\partial \alpha_k} \in L^2(\Omega_u^F)/\mathbf{R}$ verify the following system:*

$$\left\{ \begin{array}{rl} a_F(\alpha, \frac{\partial v}{\partial \alpha_k}, w) + b_F(\alpha, w, \frac{\partial p}{\partial \alpha_k}) & = c_F(\alpha, v, p), \quad \forall\, w \in (H^1(\Omega_u^F))^2 \\ b_F(\alpha, \frac{\partial v}{\partial \alpha_k}, q) & = 0 \quad \forall\, q \in L^2(\Omega_u^F)/\mathbf{R}, \end{array} \right. \tag{16}$$

Where,

$$\begin{array}{rcl} a_F(\alpha, \frac{\partial v}{\partial \alpha_k}, w) & = & \int_{\Omega_u^F} \mu \nabla \frac{\partial v}{\partial \alpha_k} \cdot \nabla w dx \\ b_F(\alpha, w, \frac{\partial p}{\partial \alpha_k}) & = & -\int_{\Omega_u^F} \nabla \cdot w \, \frac{\partial p}{\partial \alpha_k} \, dx \\ c_F(\alpha, v, p) & = & \sum_{i=1}^2 \int_{\Omega_u^F} \frac{1}{a_{11}} \frac{\partial a_{11}}{\partial \alpha_k} f_i^F w_i dx \\ & & + \int_{\Omega_u^F} \mu \left(\frac{1}{a_{11}} \frac{\partial a_{11}}{\partial \alpha_k} \frac{\partial w_1}{\partial x_1} - (\frac{1}{a_{11}} \frac{\partial a_{12}}{\partial \alpha_k} - \frac{1}{a_{11}^2} \frac{\partial a_{11}}{\partial \alpha_k} a_{12}) \frac{\partial w_1}{\partial x_2} \right) p dx \\ & & - \sum_{i=1}^2 \int_{\Omega_u^F} \mu \left(\frac{1}{a_{11}} \frac{\partial a_{11}}{\partial \alpha_k} \frac{\partial v_i}{\partial x_1} \frac{\partial w_i}{\partial x_1} - (\frac{1}{a_{11}} \frac{\partial a_{12}}{\partial \alpha_k} - \frac{1}{a_{11}^2} \frac{\partial a_{11}}{\partial \alpha_k} a_{12}) \frac{\partial v_i}{\partial x_1} \frac{\partial w_i}{\partial x_2} \right) dx \end{array} \tag{17}$$

$$a_{11} = \frac{H + u(x_1)}{H},\ a_{12} = \frac{u^{'}(x_1)}{H} x_2,\ \frac{\partial a_{11}}{\partial \alpha_k} = \frac{1}{H} \frac{\partial u(x_1, \alpha)}{\partial \alpha_k},\ \frac{\partial a_{12}}{\partial \alpha_k} = \frac{1}{H} \frac{\partial u^{'}(x_1, \alpha)}{\partial \alpha_k} x_2$$

Remark 1 *To compute the gradient, we need to solve:*

- *the structure equation (1)-(2), to find u,*
- *the fluid equation (3)-(9), to find v and p,*
- *the system (16), to find $\frac{\partial p(x_1, H+u(x_1))}{\partial \alpha_k}$.*

Remark 2 *In [13], author solves fluid-structure problem in reference domain Ω_0^F. In the paper, we solve the fluid-structure problem in the moving domain Ω_u^F.*

5 Numerical results

We assume that the velocity on the boundary fluid domain is:

$$\begin{array}{rl} v_1(x_1, x_2) & = \left\{ \begin{array}{c} 30(1 - \frac{x_2^2}{H^2}), \quad (x_1, x_2) \in \Sigma_1 \\ 30, \quad \forall (x_1, x_2) \in \Sigma_2 \end{array} \right. \\ v_2(x_1, x_2) & = 0, \quad \forall (x_1, x_2) \in \Gamma_u \end{array}$$

The parameter values of the fluid and the structure are:

Parameter related to fluid: The fluid viscosity is $\mu = 0.035\frac{g}{cm\cdot s}$, the channel length is $L = 3cm$, the channel width is $H = 0.5cm$, the volume force of the fluid $f^F = (0, 0)$.

Paramater related to structure: The structure thickness $h = 0.1cm$, Young's modulus is $E = 0.75 \cdot 10^6 \frac{g}{cm\cdot s^2}$, the moment of inertia is $I = \frac{h^3}{12}$.

We use the $P2$ Lagrange finite element to approach velocities and $P1$ Lagrange finite element is used to approach the pressure. FreeFem++ [8] is used for the numerical tests.

Table 1: Optimal values with BFGS method after 10 iterations, the starting point $\alpha_{initial} = 0$.

N	ω	J($\alpha_{initial}$)	J($\alpha_{optimal}$)	$\|J\|_\infty$	CPU
1	1	317.524	1.30957	6.70×10^{-4}	9.58s
2	1	317.524	1.30872	4.118×10^{-4}	12.29s
3	1	317.524	0.169343	6.970×10^{-3}	14.19s
4	1	317.524	0.161446	4.013×10^{-4}	17.28s
6	1	317.524	0.0406271	4.747×10^{-4}	20.44s

Table 2: $\alpha_{optimal}$ with BFGS method after 10 iterations, the starting point $\alpha_{initial} = 0$.

N	$\alpha_{optimal}$
1	(9.83274, 7.68247)
2	(9.83582, 7.67294, 0.0392021)
3	(9.76943, 7.82079, -0.105078, 1.27328)
4	(9.777, 7.80926, -0.0928571, 1.25828, 0.102329)
6	(9.75304, 7.86072, -0.141766, 1.30454, 0.0567803, 0.403188, 0.0649477)

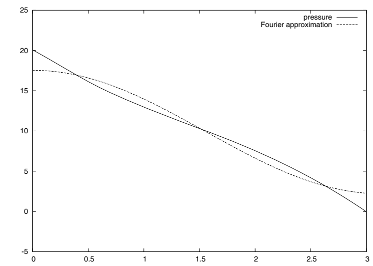

Figure 2: Pressure and approximation case N=2.

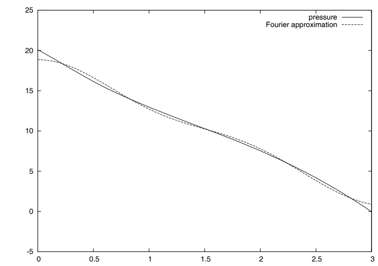

Figure 3: Pressure and approximation case N=4.

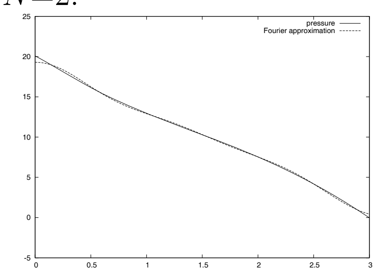

Figure 4: Pressure and approximation case N=6.

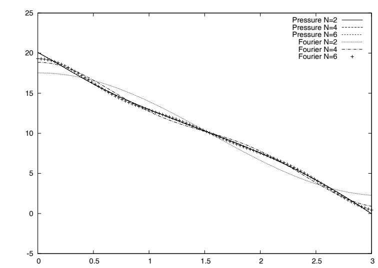

Figure 5: case N=2,4 and 6.

The above figures display the pressure on the interface and the Fourier series approximation.

When the number of Fourier coefficients N is increased, the approximation increases as well. It follows, therefore, that the approximation of the pressure on the interface by Fourier cosine series is becoming more accurate that greater the number of coefficients taken and tending to the pressure itself.

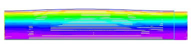

Figure 6: velocity case N=2.

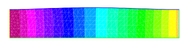

Figure 7: Pressure case N=2.

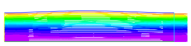

Figure 8: velocity case N=4.

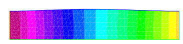

Figure 9: Pressure case N=4.

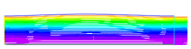

Figure 10: velocity case N=6.

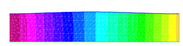

Figure 11: Pressure case N=6.

The above figures display the structure displacement, the fluid velocity and the pressure.

6 Conclusion

In this work, we introduce Fourier series development to solve fluid structure interaction problem. This approximation enable us to determine easily an analytic solution to structure equation. Using least squares method, we establish an optimization problem to compute Fourier coefficients α, the displacement u, the velocity v and the pressure p. Fourier series gives good results for N=6. It can be noted, if Fourier coefficients N is increased then J goes to zero and the coupled problem becomes equivalent to the optimization problem. This article is coming also with a view to extend and improve approximation methods of pressures on the interface.

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