Variations on Weyl Type Theorems

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Abstract. In this paper we introduce and study two new properties (Bb) and (Bab) in connection with Weyl type theorems. It is shown that if $T$ is a bounded linear operator acting on a Banach space $X$, then property (Bb) holds for $T$ if and only if generalized Browder’s theorem holds for $T$ and $\pi(T) = \pi_0(T)$, where $\pi(T)$ (resp., $\pi_0(T)$) is the set of poles of resolvent of $T$ (resp., the set of poles of resolvent of $T$ of finite rank). A similar type of result has been obtained for property (Bab).

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1 Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space $X$. For $T \in B(X)$, we denote by $T^*$, $\sigma(T)$, $\sigma_a(T)$, $N(T)$ and $R(T)$ the adjoint, the spectrum, the approximate spectrum, the null space and the range space of $T$, respectively. Let $\alpha(T)$ and $\beta(T)$
denote the dimension of the kernel \( N(T) \) and the codimension of the range \( R(T) \), respectively. Let \( E(T) \) be the set of all isolated eigen values of \( T \) and \( E^a(T) \) be the set of all eigenvalues of \( T \) which are isolated in \( \sigma_a(T) \). If the range \( R(T) \) of \( T \) is closed and \( \alpha(T) < \infty \) (resp., \( \beta(T) < \infty \)), then \( T \) is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator.

If \( T \) is either an upper or a lower semi-Fredholm then \( T \) is called a semi-Fredholm operator, while \( T \) is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If \( T \in B(X) \) is semi-Fredholm, then the index of \( T \) is defined as \( \text{ind}(T) = \alpha(T) - \beta(T) \). An operator \( T \in B(X) \) is called Weyl if it is Fredholm operator of index 0. The Weyl spectrum \( \sigma_W(T) \) of \( T \) is defined by \( \sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \} \).

For a bounded linear operator \( T \) and a nonnegative integer \( n \), we define \( T_n \) to be the restriction of \( T \) to \( R(T^n) \) viewed as a map from \( R(T^n) \) into itself (in particular \( T_0 = T \)). If for some integer \( n \), the range space \( R(T^n) \) is closed and \( T_n \) is an upper (resp., a lower) semi-Fredholm operator, then \( T \) is called an upper (resp., a lower) semi-B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. Moreover, if \( T_n \) is a Fredholm operator, then \( T \) is called a B-Fredholm operator. By Proposition 2.1 if \( T_n \) is a semi-Fredholm operator then \( T_m \) is also a semi-Fredholm operator for each \( m \geq n \) and \( \text{ind}(T_m) = \text{ind}(T_n) \). Thus the index of a semi-B-Fredholm operator \( T \) is defined as the index of the semi-Fredholm operator \( T_n \) (see [4, 5]). An operator \( T \in B(X) \) is called B-Weyl operator if it is B-Fredholm operator of index 0. The B-Weyl spectrum \( \sigma_{BW}(T) \) of \( T \) is defined as \( \sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not } \text{B-Fredholm} \} \).

Denote by \( USBF^-(X) \) the class of all upper semi B-Fredholm operators with index less than or equal to 0. Set \( \sigma_{asbf^-}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin USBF^-(X) \} \).

The descent \( q(T) \) and the ascent \( p(T) \) of \( T \) are given by \( q(T) = \inf \{ n : R(T^n) = R(T^{n+1}) \} \) and \( p(T) = \inf \{ n : N(T^n) = N(T^{n+1}) \} \). An operator \( T \in B(X) \) is said to be Drazin invertible if it has finite ascent and descent. The Drazin spectrum is defined by \( \sigma_D(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible} \} \).

Define the set \( LD(X) \) by \( LD(X) = \{ T \in B(X) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed} \} \) and \( \sigma_{LD}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not in } LD(X) \} \). Following [5], an operator \( T \in B(X) \) is said to be left Drazin invertible if \( T \in LD(X) \). We say that \( \lambda \in \sigma_a(T) \) is a left pole of \( T \) if \( T - \lambda I \in LD(X) \), and that \( \lambda \in \sigma_a(T) \) is a left pole of \( T \) of finite rank if \( \lambda \) is a left pole of \( T \) and \( \alpha(T - \lambda I) < \infty \). We denote by \( \pi^a(T) \) the set of all left poles of \( T \), and by \( \pi_0^a(T) \) the set of all left poles of finite rank. Let \( \pi(T) \) be the set of all poles of the resolvent of \( T \) and let \( \pi_0(T) \) be the set of all poles of the resolvent of finite rank. Here and elsewhere in this paper \( \sigma_{iso}(T) \) and \( \sigma_{iso}^a(T) \) denote the sets of isolated points of \( \sigma(T) \) and \( \sigma_a(T) \), respectively. Let \( E_0(T) = \{ \lambda \in \sigma_{iso}(T) : 0 < \alpha(T - \lambda I) < \infty \} \). Then we say that \( T \in B(X) \) satisfy
Variations on Weyl type theorems

(i) Weyl’s theorem if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$.

(ii) Browder’s theorem if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$.

(iii) generalized Weyl’s theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.

(iv) generalized Browder’s theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \pi(T)$.

(v) generalized a-Weyl’s theorem if $\sigma_{u a f}^{-1}(T) = \sigma_a(T) \setminus E^a(T)$.

(vi) generalized a-Browder’s theorem if $\sigma_{u a f}^{-1}(T) = \sigma_a(T) \setminus \pi^a(T)$.

The single valued extension property was introduced by Dunford ([7, 8]) and it plays an important role in local spectral theory and Fredholm theory ([1, 10]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$) if for every open disc $U$ of $\lambda_0$ the only analytic function $f : U \to X$ which satisfies the equation $(T - \lambda I) f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$.

An operator $T \in B(X)$ is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. An operator $T$ has SVEP at every isolated point of the spectrum.

Furthermore, we have $\sigma_a(T)$ does not cluster at $\lambda \Rightarrow T$ has SVEP at $\lambda$.

We say that $T \in B(X)$ satisfies property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$.

Property (Bw) has been introduced and studied in [9].

In this paper, we define and study two new properties (Bb) and (Bab). We prove that $T \in B(X)$ satisfies property (Bb) if and only if generalized Browder’s theorem holds for $T$ and $\pi(T) = \pi_0(T)$. We prove that property (Bw) implies property (Bb) but the converse is not true in general. For property (Bab) we show that if an operator $T \in B(X)$ satisfies property (Bab), then generalized a-Browder’s theorem holds for $T$ and $\sigma_{a}^{-}(T) = \sigma_{u a f}^{-}(T) \cup \sigma_{i a o}^{-}(T)$.

2 Property (Bb)

We define property (Bb) as follows:

**Definition 2.1.** A bounded linear operator $T \in B(X)$ is said to satisfy property (Bb) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T).$$

Following example provides an operator that satisfies property (Bb):
Example 2.2. Define quasinilpotent operator \( Q \in B(l^2) \) as \( Q(x_0, x_1, x_2, \ldots) = (\frac{1}{2}x_1, \frac{3}{2}x_2, \ldots) \) and let \( N \in B(l^2) \) be a nilpotent operator. Let \( T = Q \oplus N \). Then \( \sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\} \) and \( E_0(T) = \pi_0(T) = \phi \), which implies that \( T \) satisfies property \((Bb)\).

Next is example of an operator that fails to satisfy property \((Bb)\):

Example 2.3. Let \( I_1 \) be the identity on \( \mathbb{C} \). Let \( T_1 \) be defined on \( l_2 \) by

\[
T_1(x_1, x_2, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{2}x_2, \ldots).
\]

Let \( T = I_1 \oplus T_1 \). Clearly \( \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\} \cup \{1\} \). It is shown in [11] that

\[
\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\}.
\]

Thus we have that

\[
\sigma(T) \setminus \sigma_{BW}(T) = \{1\}.
\]

But \( \{1\} \) is not a pole of finite rank. Thus property \((Bb)\) is not satisfied.

Theorem 2.4. Let \( T \in B(X) \). Then \( T \) satisfies property \((Bw)\) if and only if \( T \) satisfies property \((Bb)\) and \( \pi_0(T) = E_0(T) \).

Proof. Suppose \( T \) satisfies property \((Bw)\), then \( \sigma(T) \setminus \sigma_{BW}(T) = E_0(T) \).

If \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \), then \( \lambda \in \sigma_{iso}(T) \) and \( T - \lambda I \) is B-Fredholm operator of index zero. Hence from [3, Theorem 2.3] we have \( \lambda \in \pi(T) \). Also \( \lambda \in E_0(T) \) implies that \( \dim N(T - \lambda I) < \infty \), therefore \( \lambda \in \pi_0(T) \). This implies that \( \sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi_0(T) \). Now if \( \lambda \in \pi_0(T) \), then \( \lambda \in E_0(T) \) because \( \pi_0(T) \subseteq E_0(T) \) for every \( T \in B(X) \). As \( T \) satisfies property \((Bw)\), therefore we have that \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \). Hence \( \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T) \), i.e., \( T \) satisfies property \((Bb)\) and \( \pi_0(T) = E_0(T) \). Conversely, assume that \( T \) satisfies property \((Bb)\) and \( \pi_0(T) = E_0(T) \). Then \( \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T) \), and \( \pi_0(T) = E_0(T) \). So \( \sigma(T) \setminus \sigma_{BW}(T) = E_0(T) \) and \( T \) satisfies property \((Bw)\). 

The following example shows that in general property \((Bb)\) does not imply property \((Bw)\).

Example 2.5. Let \( T \in B(l^2(N)) \) be defined by \( T(\xi_1, \xi_2, \xi_3, \ldots) = (\frac{\xi_2}{3}, \frac{\xi_3}{4}, \frac{\xi_4}{5}, \ldots) \).

Then \( \sigma(T) = \{0\} \), \( \sigma_{BW}(T) = \{0\} \), \( E_0(T) = \{0\} \) and \( \pi_0(T) = \phi \). This implies that \( \sigma(T) \setminus \sigma_{BW}(T) \neq E_0(T) \) and \( \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T) \). Therefore, \( T \) satisfies property \((Bb)\) but \( T \) does not satisfy property \((Bw)\).

Theorem 2.6. Let \( T \in B(X) \) satisfy property \((Bb)\). Then generalized Browder’s theorem holds for \( T \) and \( \sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T) \).
Proof. By [6, Proposition 3.9] it is sufficient to prove that $T$ has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Let us assume that $\lambda \notin \sigma_{BW}(T)$.

Case (i): If $\lambda \notin \sigma(T)$ then $T$ has SVEP at $\lambda$.

Case (ii): If $\lambda \in \sigma(T)$ and suppose that $T$ satisfies property (Bb) then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$ and hence $\lambda \in \sigma_{iso}(T)$, so also in this case $T$ has SVEP at $\lambda$.

We observe that $\sigma_{BW}(T) \cup \sigma_{iso}(T) \subseteq \sigma(T)$ for every $T \in B(X)$. For the reverse inclusion, let $\lambda \in \sigma(T)$ and if $\lambda \notin \sigma_{BW}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. As $T$ satisfies property (Bb), therefore $\lambda \in \pi_0(T)$. Hence $\lambda \in \sigma_{iso}(T)$, so $\sigma(T) \subseteq \sigma_{BW}(T) \cup \sigma_{iso}(T)$. Thus $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T)$.

Now we give a characterization of property (Bb) as follows:

Theorem 2.7. Let $T \in B(X)$. Then the following statements are equivalent:

(i) $T$ satisfies property (Bb),

(ii) generalized Browder’s theorem holds for $T$ and $\pi(T) = \pi_0(T)$.

Proof. (i)$\Rightarrow$(ii) Assume that $T$ satisfies property (Bb). By Theorem 2.6 generalized Browder’s theorem holds for $T$, therefore it is sufficient to prove the equality $\pi(T) = \pi_0(T)$. We know $\pi_0(T) \subseteq \pi(T)$ for all $T \in B(X)$. For the reverse inclusion let $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. Therefore the equality $\pi(T) = \pi_0(T)$.

(ii)$\Rightarrow$(i). Since generalized Browder’s theorem holds for $T$ therefore $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \pi_0(T)$.

Remark 2.8. As $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$ and $\pi(T) = \pi(T^*)$, $T$ satisfies generalized Browder’s theorem if and only if $T^*$ does.

Theorem 2.9. Let $T \in B(X)$. If $T$ or $T^*$ has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T)$, then $T$ satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$.

Proof. The hypothesis $T$ or $T^*$ has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$ implies that $T$ satisfies generalized Browder’s theorem ([6, Proposition 3.9]). By Theorem 2.7, we have $T$ satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$.

Definition 2.10. Operators $S, T \in B(X)$ are said to be injectively intertwined, denoted $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$.

If $S \prec_i T$, then $T$ has SVEP at a point $\lambda$ implies $S$ has SVEP at a point $\lambda$. To see this, let $T$ have SVEP at $\lambda$, let $U$ be an open neighbourhood of $\lambda$ and
let \( f : U \to X \) be an analytic function such that \((S - \mu)f(\mu) = 0\) for every \( \mu \in U \). Then \( U(S - \mu)f(\mu) = (T - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0 \). Since \( U \) is injective, \( f(\mu) = 0 \), i.e., \( S \) has SVEP at \( \lambda \).

**Theorem 2.11.** Let \( S, T \in B(X) \). If \( T \) has SVEP and \( S \prec_i T \), then \( S \) satisfies property (Bb) if and only if \( \pi_0(S) = \pi(S) \).

**Proof.** Suppose that \( T \) has SVEP. Since \( S \prec_i T \), therefore \( S \) has SVEP. Hence the result follows from Theorem 2.9.

**Definition 2.12.** An operator \( T \in B(X) \) is said to be finitely polaroid (resp., polaroid) if all the isolated points of its spectrum are poles of finite rank, i.e., \( \sigma_{iso}(T) \subseteq \pi_0(T) \), (resp., \( \sigma_{iso}(T) \subseteq \pi(T) \)).

Finitely polaroid operators are polaroid but not conversely. In the following result, we give a condition under which the converse is also true.

**Theorem 2.13.** A polaroid operator satisfying property (Bb) is finitely polaroid.

**Proof.** Let \( T \in B(X) \) be a polaroid operator satisfying property (Bb), then the result follows from Theorem 2.7.

**Theorem 2.14.** Let \( T \in B(X) \) be finitely polaroid. Then \( T \) satisfies property (Bb) if

(i) \( T \) satisfies generalized Weyl’s theorem, or

(ii) \( T \) or \( T^* \) has SVEP.

**Proof.** In both the cases \( T \) satisfies generalized Browder’s theorem [3, Corollary 2.6] and [2, Theorem 2.3]. Suppose \( \lambda \in \pi(T) \), then \( \lambda \) is isolated in \( \sigma(T) \), i.e., \( \lambda \in \sigma_{iso}(T) \subset \pi_0(T) \), as \( T \) is finitely polaroid. Other inclusion is always true. Thus \( \pi(T) = \pi_0(T) \). From Theorem 2.7, we have that \( T \) satisfies property (Bb).

**Theorem 2.15.** Let \( T \in B(X) \) be a polaroid operator and satisfy property (Bb). Then generalized Weyl’s theorem holds for \( T \).

**Proof.** \( T \) is polaroid and satisfies property (Bb) \( \Leftrightarrow \).

\[ \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T) \subseteq E(T) = \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) \] (Since \( T \) satisfies generalized Browder’s theorem by Theorem 2.6).

The condition of polaroid operator cannot be dropped from the Theorem 2.15. The following example shows that theorem may not hold if the operator is not a polaroid operator.
Example 2.16. Let $X$ be an infinite dimensional Banach space. Let $T \in B(X)$ be a nilpotent operator such that $R(T)$ is not closed and $Q \in B(X)$ be a quasinilpotent operator which is not nilpotent. Let $S = T \oplus Q \in B(X \oplus X)$. Then $\sigma(S) = \sigma_W(S) = \sigma_{BW}(S) = E(S) = \{0\}$, $E_0(S) = \pi_0(S) = \pi(S) = \emptyset$. Property (Bb) is satisfied but generalized Weyl's theorem is not satisfied as the operator is not a polaroid operator.

Remark 2.17. By Theorem 2.14 and Theorem 2.15, we have in the case of finitely polaroid operators, $T$ satisfies property (Bb) is equivalent to $T$ satisfies generalized Weyl's theorem.

Definition 2.18. The analytic core of an operator $T \in B(X)$ is the subspace $K(T)$ of all $x \in X$ such that there exists a sequence $\{x_n\}$ and a constant $c > 0$ such that

(i) $Tx_{n+1} = x_n$, $x = x_0$

(ii) $\|x_n\| \leq c^n\|x\|$ for $n = 1, 2, \ldots ,

For every $T \in B(X)$, we have $\sigma_{BW}(T) \subseteq \sigma_W(T)$. Hence, if $T$ satisfies property (Bb), then $\sigma(T) \setminus \sigma_W(T) \subseteq \sigma(W) \setminus \sigma_{BW}(T) = \pi_0(T)$. Thus, if $\sigma_{iso}(T) = \emptyset$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ (and $T$ satisfies Weyl's theorem and generalized Weyl's theorem). For a non-quasinilpotent operator $T \in B(X)$, a condition guaranteeing $\sigma_{iso}(T) = \emptyset$ is that $K(T) = \{0\}$.

Theorem 2.19. Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ and $T$ satisfies both property (Bb) and generalized Weyl's theorem.

Proof. If $T \in B(X)$ is not quasinilpotent and $K(T) = \{0\}$, then $T$ has SVEP, $\sigma(T) = \sigma_W(T)$ is a connected set containing 0 and $\sigma_{iso}(T) = \emptyset$ [1, Theorem 3.121]. SVEP implies $T$ satisfies generalized Browder’s theorem. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \emptyset = \pi_0(T) = E_0(T) = E(T)$, i.e., $T$ satisfies property (Bb) and generalized Weyl’s theorem (so also Weyl’s theorem).

Remark 2.20. If $T \in B(X)$ is quasinilpotent, then $\sigma(T) = \sigma_{BW}(T) = \{0\}$; hence $T$ satisfies property (Bb) is equivalent to $T$ satisfies Browder’s theorem.

3 Property (Bab)

Definition 3.1. A bounded linear operator $T \in B(X)$ is said to satisfy property (Bab) if $\sigma_a(T) \setminus \sigma_{abf}^{-}(T) = \pi_0(T)$.

Example 3.2. Let $R \in B(l^2(\mathbb{N}))$ be the unilateral right shift and $P \in B(l^2(\mathbb{N}))$ be the operator defined by $P(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, x_4, \ldots)$. Consider $T = R \oplus P$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, then $\sigma(T) = D(0, 1)$ is the closed unit disc in $\mathbb{C}$,
so that $\sigma(T)$ has no isolated point. Thus $\sigma_{\text{iso}}(T) = \emptyset$. Furthermore $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle in $\mathbb{C}$ and $\sigma_{\text{usbf}}(T) = C(0,1)$. This implies that $\sigma_a(T) \setminus \sigma_{\text{usbf}}(T) = \{0\}$. Moreover, we have $\pi_0^a(T) = \{0\}$. Therefore $T$ satisfies property (Bab).

**Example 3.3.** Let $R$ be the unilateral right shift operator defined on $l^2(\mathbb{N})$ and $T$ be the operator defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by $T = 0 \oplus R$, then $\sigma(T) = D(0,1)$ and $\sigma_a(T) = \{0\} \cup C(0,1)$. Now $\{0\} \notin \sigma_{\text{usbf}}(T)$. Therefore $T$ does not satisfy property (Bab), since $\{0\} \in \sigma_a(T) \setminus \sigma_{\text{usbf}}(T)$ and $\pi_0^a(T) = \emptyset$.

**Theorem 3.4.** Let $T \in B(X)$ satisfy property (Bab). Then generalized a-Browder’s theorem holds for $T$ and $\sigma_a(T) = \sigma_{\text{usbf}}(T) \cup \sigma_{\text{iso}}^a(T)$.

**Proof.** By Proposition 3.10 of [6] it is sufficient to prove that $T$ has SVEP at every $\lambda \notin \sigma_{\text{usbf}}(T)$. Let us assume that $\lambda \notin \sigma_{\text{usbf}}(T)$.

If $\lambda \notin \sigma_a(T)$ then $T$ has SVEP at $\lambda$. If $\lambda \in \sigma_a(T)$ and suppose that $T$ satisfies property (Bab) then $\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}}(T) = \pi_0^a(T)$. Thus $\lambda \in \sigma_{\text{iso}}^a(T)$ which implies $T$ has SVEP at $\lambda$. To prove that $\sigma_a(T) = \sigma_{\text{usbf}}(T) \cup \sigma_{\text{iso}}^a(T)$. We observe that $\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}}(T) = \pi_0^a(T)$. Thus $\lambda \in \sigma_{\text{iso}}^a(T)$. Hence $\sigma_a(T) \subseteq \sigma_{\text{usbf}}(T) \cup \sigma_{\text{iso}}^a(T)$. But $\sigma_{\text{usbf}}(T) \cup \sigma_{\text{iso}}^a(T) \subseteq \sigma_a(T)$ for every $T \in B(X)$. Thus $\sigma_a(T) = \sigma_{\text{usbf}}(T) \cup \sigma_{\text{iso}}^a(T)$.

A characterization of property (Bab) is as follows:

**Theorem 3.5.** Let $T \in B(X)$. Then the following statements are equivalent:

(i) $T$ satisfies property (Bab);

(ii) generalized a-Browder’s theorem holds for $T$ and $\pi^a(T) = \pi_0^a(T)$.

**Proof.** (i)$\Rightarrow$(ii) Assume that $T$ satisfies property (Bab).

By Theorem 3.4 it is sufficient to prove the equality $\pi^a(T) = \pi_0^a(T)$. If $\lambda \in \pi^a(T)$ then as $T$ satisfies generalized a-Browder’s theorem, it implies that $\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}}(T) = \pi_0^a(T)$, because $T$ satisfies property (Bab). Other inclusion is always true. Therefore the equality $\pi^a(T) = \pi_0^a(T)$.

(ii)$\Rightarrow$(i). If $\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}}(T)$, then generalized a-Browder’s theorem implies that $\lambda \in \pi^a(T) = \pi_0^a(T)$. Conversely, if $\lambda \in \pi_0^a(T)$ then $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{\text{usbf}}(T)$. Thus $\pi_0^a(T) = \sigma_a(T) \setminus \sigma_{\text{usbf}}(T)$.

**Remark 3.6.** Property (Bab) implies property (Bb), but the converse is not true in general. Indeed, if we consider the operator defined in Example 3.3, then $T$ does not satisfy property (Bab). Note that $T$ satisfies property (Bb), since $\sigma_{\text{BW}}(T) = D(0,1)$ and $\pi_0(T) = \emptyset$.

**Theorem 3.7.** Let $T \in B(X)$. If $T$ has SVEP at points in $\sigma_a(T) \setminus \sigma_{\text{usbf}}(T)$, then $T$ satisfies property (Bab) if and only if $\pi_0^a(T) = \pi^a(T)$.
Proof. The hypothesis \( T \) has SVEP at points in \( \sigma_a(T) \setminus \sigma_{usbf^{-}}(T) \) implies that \( T \) satisfies generalized a-Browder’s theorem \([6, \text{Proposition 3.10}]\). Hence, if \( \pi^a(T) = \pi^a(T) \), then \( \sigma_a(T) \setminus \sigma_{usbf^{-}}(T) = \pi^a(T) = \pi^a_{0}(T) \). □

**Theorem 3.8.** Let \( S, T \in B(X) \). If \( T \) has SVEP and \( S \prec_i T \), then \( S \) satisfies property (Bab) if and only if \( \pi^a(S) = \pi^a(T) \).

**Proof.** Suppose that \( T \) has SVEP. Since \( S \prec_i T \), therefore \( S \) has SVEP. Hence the result follows from Theorem 3.7. □

**Definition 3.9.** An operator \( T \in B(X) \) is said to be finitely left-polaroid (resp., left-polaroid) if all the isolated points of its approximate spectrum are left poles of finite rank, i.e., \( \sigma^a_{iso}(T) \subseteq \pi^a_{0}(T) \) (resp., \( \sigma^a_{iso}(T) \subseteq \pi^a(T) \)).

Finitely left-polaroid operators are left-polaroid but the converse is not true in general. In the next result we give a condition under which the converse is also true.

**Theorem 3.10.** A left-polaroid operator satisfying property (Bab) is finitely left-polaroid.

**Proof.** Let \( T \in B(X) \) be a left-polaroid operator satisfying property (Bab). Then Theorem 3.5 implies that \( T \) is finitely left-polaroid. □

**Theorem 3.11.** Let \( T \in B(X) \) be finitely left-polaroid. Then \( T \) satisfies property (Bab) if \( T \) satisfies generalized a-Weyl’s theorem.

**Proof.** From \([4, \text{Corollary 3.3}]\), we have that \( T \) satisfies generalized a-Browder’s theorem. Suppose \( \lambda \in \pi^a(T) \), then \( \lambda \) is isolated in \( \sigma_a(T) \), i.e., \( \lambda \in \sigma^a_{iso}(T) \subseteq \pi^a_{0}(T) \) (resp., \( \sigma^a_{iso}(T) \subseteq \pi^a(T) \)). As \( T \) is finitely left-polaroid, other inclusion is always true. Thus \( \pi^a(T) = \pi^a_{0}(T) \). By Theorem 3.5, we have that \( T \) satisfies property (Bab). □

**Theorem 3.12.** Let \( T \in B(X) \) be left-polaroid and satisfy property (Bab). Then generalized a-Weyl’s theorem holds for \( T \).

**Proof.** \( T \) is left-polaroid and satisfies property (Bab) \( \Leftrightarrow \)

\[
\sigma_a(T) \setminus \sigma_{usbf^{-}}(T) = \pi^a_{0}(T) \subseteq \pi^a(T) = \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf^{-}}(T)
\]

(Since \( T \) satisfies generalized a-Browder’s theorem by Theorem 3.5). □

**Remark 3.13.** By Theorem 3.11 and Theorem 3.12, finitely left-polaroid operator \( T \) satisfies property (Bab) is equivalent to \( T \) satisfies generalized a-Weyl’s theorem.

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References


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