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Variations on Weyl Type Theorems

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Abstract. In this paper we introduce and study two new properties (Bb) and (Bab) in connection with Weyl type theorems. It is shown that if T is a bounded linear operator acting on a Banach space X, then property (Bb) holds for T if and only if generalized Browder's theorem holds for T and $\pi(T) = \pi_0(T)$, where $\pi(T)$ (resp., $\pi_0(T)$) is the set of poles of resolvent of T (resp., the set of poles of resolvent of T of finite rank). A similar type of result has been obtained for property (Bab).

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1 Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For $T \in B(X)$, we denote by T^*, $\sigma(T)$, $\sigma_a(T)$, $N(T)$ and $R(T)$ the adjoint, the spectrum, the approximate spectrum, the null space and the range space of T, respectively. Let $\alpha(T)$ and $\beta(T)$

denote the dimension of the kernel $N(T)$ and the codimension of the range $R(T)$, respectively. Let $E(T)$ be the set of all isolated eigen values of T and $E^a(T)$ be the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator.

If T is either an upper or a lower semi-Fredholm then T is called a semi-Fredholm operator, while T is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If $T \in B(X)$ is semi-Fredholm, then the index of T is defined as $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in B(X)$ is called Weyl if it is Fredholm operator of index 0. The Weyl spectrum $\sigma_W(T)$ of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$.

For a bounded linear operator T and a nonnegative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi-B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. ¿From [5, Proposition 2.1] if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$ and $\text{ind}(T_m) = \text{ind}(T_n)$. Thus the index of a semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator T_n (see [4, 5]). An operator $T \in B(X)$ is called B-Weyl operator if it is B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl operator}\}$. Denote by $USBF^{-}(X)$ the class of all upper semi B-Fredholm operators with index less than or equal to 0. Set $\sigma_{usbf^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin USBF^{-}(X)\}$.

The descent $q(T)$ and the ascent $p(T)$ of T are given by $q(T) = \inf\{n : R(T^n) = R(T^{n+1})\}$ and $p(T) = \inf\{n : N(T^n) = N(T^{n+1})\}$. An operator $T \in B(X)$ is said to be Drazin invertible if it has finite ascent and descent. The Drazin spectrum is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$.

Define the set $LD(X)$ by $LD(X) = \{T \in B(X) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } LD(X)\}$. Following [5], an operator $T \in B(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. We denote by $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles of finite rank. Let $\pi(T)$ be the set of all poles of the resolvent of T and let $\pi_0(T)$ be the set of all poles of the resolvent of finite rank. Here and elsewhere in this paper $\sigma_{\text{iso}}(T)$ and $\sigma_a^{\text{iso}}(T)$ denote the sets of isolated points of $\sigma(T)$ and $\sigma_a(T)$, respectively. Let $E_0(T) = \{\lambda \in \sigma_{\text{iso}}(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Then we say that $T \in B(X)$ satisfy

(i) Weyl's theorem if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$.

(ii) Browder's theorem if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$.

(iii) generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.

(iv) generalized Browder's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \pi(T)$.

(v) generalized a-Weyl's theorem if $\sigma_{usbf^-}(T) = \sigma_a(T) \setminus E^a(T)$.

(vi) generalized a-Browder's theorem if $\sigma_{usbf^-}(T) = \sigma_a(T) \setminus \pi^a(T)$.

The single valued extension property was introduced by Dunford ([7, 8]) and it plays an important role in local spectral theory and Fredholm theory ([1, 10]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$, is the function $f \equiv 0$.

An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. An operator T has SVEP at every isolated point of the spectrum.

Furthermore, we have
$\sigma_a(T)$ does not cluster at $\lambda \Rightarrow T$ has SVEP at λ.
We say that $T \in B(X)$ satisfies property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$. Property (Bw) has been introduced and studied in [9].

In this paper, we define and study two new properties (Bb) and (Bab). We prove that $T \in B(X)$ satisfies property (Bb) if and only if generalized Browder's theorem holds for T and $\pi(T) = \pi_0(T)$. We prove that property (Bw) implies property (Bb) but the converse is not true in general. For property (Bab) we show that if an operator $T \in B(X)$ satisfies property (Bab), then generalized a-Browder's theorem holds for T and $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T)$.

2 Property (Bb)

We define property (Bb) as follows:

Definition 2.1. A bounded linear operator $T \in B(X)$ is said to satisfy property (Bb) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)\,.$$

Following example provides an operator that satisfies property (Bb):

Example 2.2. Define quasinilpotent operator $Q \in B(l^2)$ as $Q(x_0, x_1, x_2, \dots) = (\frac{1}{2}x_1, \frac{1}{3}x_2, \dots)$ and let $N \in B(l^2)$ be a nilpotent operator. Let $T = Q \oplus N$. Then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\}$ and $E_0(T) = \pi_0(T) = \phi$, which implies that T satisfies property (Bb).

Next is example of an operator that fails to satisfy property (Bb):

Example 2.3. Let I_1 be the identity on $\mathbb{C}$. Let T_1 be defined on l_2 by

$$T_1(x_1, x_2, \dots) = (0, \tfrac{1}{3}x_1, \tfrac{1}{3}x_2, \dots)\,.$$

Let $T = I_1 \oplus T_1$. Clearly $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\} \cup \{1\}$. It is shown in [11] that

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \tfrac{1}{3}\}\,.$$

Thus we have that

$$\sigma(T) \setminus \sigma_{BW}(T) = \{1\}\,.$$

But $\{1\}$ is not a pole of finite rank. Thus property (Bb) is not satisfied.

Theorem 2.4. *Let $T \in B(X)$. Then T satisfies property (Bw) if and only if T satisfies property (Bb) and $\pi_0(T) = E_0(T)$.*

Proof. Suppose T satisfies property (Bw), then $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$.

If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda \in \sigma_{\text{iso}}(T)$ and $T - \lambda I$ is B-Fredholm operator of index zero. Hence from [3, Theorem 2.3] we have $\lambda \in \pi(T)$. Also $\lambda \in E_0(T)$ implies that $\dim N(T - \lambda I) < \infty$, therefore $\lambda \in \pi_0(T)$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi_0(T)$. Now if $\lambda \in \pi_0(T)$, then $\lambda \in E_0(T)$ because $\pi_0(T) \subseteq E_0(T)$ for every $T \in B(X)$. As T satisfies property (Bw), therefore we have that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, i.e., T satisfies property (Bb) and $\pi_0(T) = E_0(T)$. Conversely, assume that T satisfies property (Bb) and $\pi_0(T) = E_0(T)$. Then $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$, and $\pi_0(T) = E_0(T)$. So $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$ and T satisfies property (Bw). □

The following example shows that in general property (Bb) does not imply property (Bw).

Example 2.5. Let $T \in B(l^2(\mathbb{N}))$ be defined by $T(\xi_1, \xi_2, \xi_3, \dots) = (\frac{\xi_2}{3}, \frac{\xi_3}{4}, \frac{\xi_4}{5}, \dots)$. Then $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$, $E_0(T) = \{0\}$ and $\pi_0(T) = \phi$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) \neq E_0(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. Therefore, T satisfies property (Bb) but T does not satisfy property (Bw).

Theorem 2.6. *Let $T \in B(X)$ satisfy property (Bb). Then generalized Browder's theorem holds for T and $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{\text{iso}}(T)$.*

Proof. By [6, Proposition 3.9] it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Let us assume that $\lambda \notin \sigma_{BW}(T)$.

Case (i): If $\lambda \notin \sigma(T)$ then T has SVEP at λ.

Case (ii): If $\lambda \in \sigma(T)$ and suppose that T satisfies property (Bb) then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$ and hence $\lambda \in \sigma_{\mathrm{iso}}(T)$, so also in this case T has SVEP at λ.

We observe that $\sigma_{BW}(T) \cup \sigma_{\mathrm{iso}}(T) \subseteq \sigma(T)$ for every $T \in B(X)$. For the reverse inclusion, let $\lambda \in \sigma(T)$ and if $\lambda \notin \sigma_{BW}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. As T satisfies property (Bb), therefore $\lambda \in \pi_0(T)$. Hence $\lambda \in \sigma_{\mathrm{iso}}(T)$, so $\sigma(T) \subseteq \sigma_{BW}(T) \cup \sigma_{\mathrm{iso}}(T)$. Thus $\sigma(T) = \sigma_{BW}(T) \cup \sigma_{\mathrm{iso}}(T)$. □

Now we give a characterization of property (Bb) as follows:

Theorem 2.7. *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) *T satisfies property (Bb),*

(ii) *generalized Browder's theorem holds for T and $\pi(T) = \pi_0(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (Bb). By Theorem 2.6 generalized Browder's theorem holds for T, therefore it is sufficient to prove the equality $\pi(T) = \pi_0(T)$. We know $\pi_0(T) \subseteq \pi(T)$ for all $T \in B(X)$. For the reverse inclusion let $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. Therefore the equality $\pi(T) = \pi_0(T)$.

(ii)⇒(i). Since generalized Browder's theorem holds for T therefore $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \pi_0(T)$. □

Remark 2.8. As $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$ and $\pi(T) = \pi(T^*)$, T satisfies generalized Browder's theorem if and only if T^* does.

Theorem 2.9. *Let $T \in B(X)$. If T or T^* has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$.*

Proof. The hypothesis T or T^* has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$ implies that T satisfies generalized Browder's theorem ([6, Proposition 3.9]). By Theorem 2.7, we have T satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$. □

Definition 2.10. Operators $S, T \in B(X)$ are said to be injectively intertwined, denoted $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$.

If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at a point λ. To see this, let T have SVEP at λ, let $\mathcal{U}$ be an open neighbourhood of λ and let $f : \mathcal{U} \to X$ be an analytic function such that $(S-\mu)f(\mu) = 0$ for every $\mu \in \mathcal{U}$. Then $U(S-\mu)f(\mu) = (T-\mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. Since U is injective, $f(\mu) = 0$, i.e., S has SVEP at λ.

Theorem 2.11. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$, then S satisfies property (Bb) if and only if $\pi_0(S) = \pi(S)$.*

Proof. Suppose that T has SVEP. Since $S \prec_i T$, therefore S has SVEP. Hence the result follows from Theorem 2.9. □

Definition 2.12. An operator $T \in B(X)$ is said to be finitely polaroid (resp., polaroid) if all the isolated points of its spectrum are poles of finite rank, i.e., $\sigma_{\text{iso}}(T) \subseteq \pi_0(T)$, (resp., $\sigma_{\text{iso}}(T) \subseteq \pi(T)$).

Finitely polaroid operators are polaroid but not conversely. In the following result, we give a condition under which the converse is also true.

Theorem 2.13. *A polaroid operator satisfying property (Bb) is finitely polaroid.*

Proof. Let $T \in B(X)$ be a polaroid operator satisfying property (Bb),then the result follows from Theorem 2.7. □

Theorem 2.14. *Let $T \in B(X)$ be finitely polaroid. Then T satisfies property (Bb) if*

(i) *T satisfies generalized Weyl's theorem, or*

(ii) *T or T^* has SVEP.*

Proof. In both the cases T satisfies generalized Browder's theorem [3, Corollary 2.6] and [2, Theorem 2.3]. Suppose $\lambda \in \pi(T)$, then λ is isolated in $\sigma(T)$, i.e., $\lambda \in \sigma_{\text{iso}}(T) \subset \pi_0(T)$, as T is finitely polaroid. Other inclusion is always true. Thus $\pi(T) = \pi_0(T)$. From Theorem 2.7, we have that T satisfies property (Bb). □

Theorem 2.15. *Let $T \in B(X)$ be a polaroid operator and satisfy property (Bb). Then generalized Weyl's theorem holds for T.*

Proof. T is polaroid and satisfies property (Bb) $\Leftrightarrow$.

$\sigma(T)\setminus\sigma_{BW}(T) = \pi_0(T) \subseteq E(T) = \pi(T) = \sigma(T)\setminus\sigma_{BW}(T)$ (Since T satisfies generalized Browder's theorem by Theorem 2.6). □

The condition of polaroid operator cannot be dropped from the Theorem 2.15. The following example shows that theorem may not hold if the operator is not a polaroid operator. □

Example 2.16. Let X be an infinite dimensional Banach space. Let $T \in B(X)$ be a nilpotent operator such that $R(T)$ is not closed and $Q \in B(X)$ be a quasinilpotent operator which is not nilpotent. Let $S = T \oplus Q \in B(X \oplus X)$. Then $\sigma(S) = \sigma_W(S) = \sigma_{BW}(S) = E(S) = \{0\}$, $E_0(S) = \pi_0(S) = \pi(S) = \phi$. Property (Bb) is satisfied but generalized Weyl's theorem is not satisfied as the operator is not a polaroid operator.

Remark 2.17. By Theorem 2.14 and Theorem 2.15, we have in the case of finitely polaroid operators, T satisfies property (Bb) is equivalent to T satisfies generalized Weyl's theorem.

Definition 2.18. The analytic core of an operator $T \in B(X)$ is the subspace $K(T)$ of all $x \in X$ such that there exists a sequence $\{x_n\}$ and a constant $c > 0$ such that

(i) $Tx_{n+1} = x_n$, $x = x_0$

(ii) $\|x_n\| \leq c^n \|x\|$ for $n = 1, 2, \ldots$.

For every $T \in B(X)$, we have $\sigma_{BW}(T) \subseteq \sigma_W(T)$. Hence, if T satisfies property (Bb), then $\sigma(T) \setminus \sigma_W(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. Thus, if $\sigma_{\text{iso}}(T) = \phi$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ (and T satisfies Weyl's theorem and generalized Weyl's theorem). For a non-quasinilpotent operator $T \in B(X)$, a condition guaranteeing $\sigma_{\text{iso}}(T) = \phi$ is that $K(T) = \{0\}$.

Theorem 2.19. *Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ and T satisfies both property (Bb) and generalized Weyl's theorem.*

Proof. If $T \in B(X)$ is not quasinilpotent and $K(T) = \{0\}$, then T has SVEP, $\sigma(T) = \sigma_W(T)$ is a connected set containing 0 and $\sigma_{\text{iso}}(T) = \phi$ [1, Theorem 3.121]. SVEP implies T satisfies generalized Browder's theorem. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \phi = \pi_0(T) = E_0(T) = E(T)$, i.e., T satisfies property (Bb) and generalized Weyl's therorem (so also Weyl's theorem). □

Remark 2.20. If $T \in B(X)$ is quasinilpotent, then $\sigma(T) = \sigma_{BW}(T) = \{0\}$; hence T satisfies property (Bb) is equivalent to T satisfies Browder's theorem.

3 Property (Bab)

Definition 3.1. A bounded linear operator $T \in B(X)$ is said to satisfy property (Bab) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T)$.

Example 3.2. Let $R \in B(l^2(\mathbb{N}))$ be the unilateral right shift and $P \in B(l^2(\mathbb{N}))$ be the operator defined by $P(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, x_4, \ldots)$. Consider $T = R \oplus P$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, then $\sigma(T) = D(0, 1)$ is the closed unit disc in $\mathbb{C}$, so that $\sigma(T)$ has no isolated point. Thus $\sigma_{\text{iso}}(T) = \phi$. Furthermore $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle in $\mathbb{C}$ and $\sigma_{usbf^-}(T) = C(0,1)$. This implies that $\sigma_a(T) \setminus_{usbf^-} (T) = \{0\}$. Moreover, we have $\pi_0^a(T) = \{0\}$. Therefore T satisfies property (Bab).

Example 3.3. Let R be the unilateral right shift operator defined on $l^2(\mathbb{N})$ and T be the operator defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by $T = 0 \oplus R$, then $\sigma(T) = D(0,1)$ and $\sigma_a(T) = \{0\} \cup C(0,1)$. Now $\{0\} \notin \sigma_{usbf^-}(T)$. Therefore T does not satisfy property (Bab), since $\{0\} \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$ and $\pi_0^a(T) = \phi$.

Theorem 3.4. *Let $T \in B(X)$ satisfy property (Bab). Then generalized a-Browder's theorem holds for T and $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T)$.*

Proof. By Proposition 3.10 of [6] it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{usbf^-}(T)$. Let us assume that $\lambda \notin \sigma_{usbf^-}(T)$.

If $\lambda \notin \sigma_a(T)$ then T has SVEP at λ. If $\lambda \in \sigma_a(T)$ and suppose that T satisfies property (Bab) then $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T)$. Thus $\lambda \in \sigma_a^{\text{iso}}(T)$ which implies T has SVEP at λ. To prove that $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T)$. We observe that $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T)$. Thus $\lambda \in \sigma_a^{\text{iso}}(T)$. Hence $\sigma_a(T) \subseteq \sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T)$. But $\sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T) \subseteq \sigma_a(T)$ for every $T \in B(X)$. Thus $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\text{iso}}(T)$. □

A characterization of property (Bab) is as follows:

Theorem 3.5. *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) *T satisfies property (Bab);*

(ii) *generalized a-Browder's theorem holds for T and $\pi^a(T) = \pi_0^a(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (Bab).

By Theorem 3.4 it is sufficient to prove the equality $\pi^a(T) = \pi_0^a(T)$. If $\lambda \in \pi^a(T)$ then as T satisfies generalized a-Browder's theorem, it implies that $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T)$, because T satisfies property (Bab). Other inclusion is always true. Therefore the equality $\pi^a(T) = \pi_0^a(T)$.

(ii)⇒(i). If $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then generalized a-Browder's theorem implies that $\lambda \in \pi^a(T) = \pi_0^a(T)$. Conversely, if $\lambda \in \pi_0^a(T)$ then $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. Thus $\pi_0^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. □

Remark 3.6. Property (Bab) implies property (Bb), but the converse is not true in general. Indeed, if we consider the operator defined in Example 3.3, then T does not satisfy property (Bab). Note that T satisfies property (Bb), since $\sigma_{BW}(T) = D(0,1)$ and $\pi_0(T) = \phi$.

Theorem 3.7. *Let $T \in B(X)$. If T has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then T satisfies property (Bab) if and only if $\pi_0^a(T) = \pi^a(T)$.*

Proof. The hypothesis T has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf^-}(T)$ implies that T satisfies generalized a-Browder's theorem [6, Proposition 3.10]. Hence, if $\pi_0^a(T) = \pi^a(T)$, then $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi^a(T) = \pi_0^a(T)$. □

Theorem 3.8. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$, then S satisfies property (Bab) if and only if $\pi_0^a(S) = \pi^a(S)$.*

Proof. Suppose that T has SVEP. Since $S \prec_i T$, therefore S has SVEP. Hence the result follows from Theorem 3.7. □

Definition 3.9. An operator $T \in B(X)$ is said to be finitely left-polaroid (resp., left-polaroid) if all the isolated points of its approximate spectrum are left poles of finite rank, i.e., $\sigma_a^{\text{iso}}(T) \subseteq \pi_0^a(T)$ (resp., $\sigma_a^{\text{iso}}(T) \subseteq \pi^a(T)$).

Finitely left-polaroid operators are left-polaroid but the converse is not true in general. In the next result we give a condition under which the converse is also true.

Theorem 3.10. *A left-polaroid operator satisfying property (Bab) is finitely left-polaroid.*

Proof. Let $T \in B(X)$ be a left-polaroid operator satisfying property (Bab). Then Theorem 3.5 implies that T is finitely left-polaroid. □

Theorem 3.11. *Let $T \in B(X)$ be finitely left-polaroid. Then T satisfies property (Bab) if T satisfies generalized a-Weyl's theorem.*

Proof. ¿From [4, Corollary 3.3], we have that T satisfies generalized a-Browder's theorem. Suppose $\lambda \in \pi^a(T)$, then λ is isolated in $\sigma_a(T)$, i.e., $\lambda \in \sigma_{\text{iso}}^a(T) \subseteq \pi_0^a(T)$, as T is finitely left-polaroid. Other inclusion is always true. Thus $\pi^a(T) = \pi_0^a(T)$. By Theorem 3.5, we have that T satisfies property (Bab). □

Theorem 3.12. *Let $T \in B(X)$ be left-polaroid and satisfy property (Bab). Then generalized a-Weyl's theorem holds for T.*

Proof. T is left-polaroid and satisfies property (Bab) $\Leftrightarrow$.

$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T) \subseteq E^a(T) = \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$ (Since T satisfies generalized a-Browder's theorem by Theorem 3.5). □

Remark 3.13. By Theorem 3.11 and Theorem 3.12, finitely left-polaroid operator T satisfies property (Bab) is equivalent to T satisfies generalized a-Weyl's theorem.

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