Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 2, 61 - 67
HIKARI Ltd, www.m-hikari.com

Sums of Perfect Powers

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina
jakimczu@mail.unlu.edu.ar

Abstract

Let P_n be the n-th perfect power. In this article we obtain asymptotic formulae for the sum $\sum_{i=1}^{n} P_i$. We also prove the following formulae

$$\sum_{i=1}^{n} \frac{1}{\sqrt{P_i}} = \log n + C + o(1), \qquad \sum_{P_n \leq x} \frac{1}{\sqrt{P_n}} = \frac{1}{2} \log x + C + o(1),$$

where C is a constant.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Perfect powers, sums of perfect powers

1 Introduction and Preliminary Results

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

Let P_n be the n-th perfect power. Jakimczuk [1] proved the following asymptotic formula

$$P_n \sim n^2. \tag{1}$$

Equation (1) implies, as is well-known, that the series $\sum_{i=1}^{\infty} \frac{1}{P_i}$ converges. Equation (1) gives

$$\sqrt{P_n} \sim n. \tag{2}$$

Equation (2) implies that the series $\sum_{i=1}^{\infty} \frac{1}{\sqrt{P_i}}$ diverges. In the next section we study this divergent series.

A quadratfrei number is a number without square factors, a product of different primes. The first few terms of the integer sequence of quadratfrei numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \ldots$$

On the other hand, the Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, if n is the product of r different primes, then $\mu(n) = (-1)^r$, if the square of a prime divides n, then $\mu(n) = 0$.

Jakimczuk [1] proved the following more strong theorem.

Theorem 1.1 *Let p_h be the h-th prime with $h \geq 3$, where h is an arbitrary but fixed positive integer. We have*

$$P_n = n^2 + \frac{13}{3} n^{8/6} + \frac{32}{15} n^{32/30} + \sum_{\substack{2 \leq q \leq p_h \\ q \neq 2,6,30}} 2\mu(q) n^{1+\frac{2}{q}} + o\left(n^{1+\frac{2}{p_h}}\right) \qquad (3)$$

where q denotes a quadratfrei number. Note that $2 = 1 + \frac{2}{2}$, $\frac{8}{6} = 1 + \frac{2}{6}$ and $\frac{32}{30} = 1 + \frac{2}{30}$.

For example, if $h = 4$ then Theorem 1.1 becomes

$$P_n = n^2 - 2n^{\frac{5}{3}} - 2n^{\frac{7}{5}} + \frac{13}{3} n^{\frac{4}{3}} - 2n^{\frac{9}{7}} + o\left(n^{\frac{9}{7}}\right). \qquad (4)$$

Using Theorem 1.1 in the next section we obtain asymptotic formulae for the sum $\sum_{i=1}^{n} P_i$. Finally, in the next section, we prove a general theorem and as corollary we obtain the following limit

$$\lim_{n \to \infty} \frac{\sqrt[n]{P_1 \cdots P_n}}{P_n} = \frac{1}{e^2}.$$

2 Main Results

Theorem 2.1 *Let p_h be the h-th prime with $h \geq 3$, where h is an arbitrary but fixed positive integer. We have*

$$\sum_{i=1}^{n} P_i = \frac{n^3}{3} + \frac{13}{7} n^{14/6} + \frac{32}{31} n^{62/30} + \sum_{\substack{2 \leq q \leq p_h \\ q \neq 2,6,30}} \mu(q) \frac{2}{2 + \frac{2}{q}} n^{2+\frac{2}{q}} + o\left(n^{2+\frac{2}{p_h}}\right) \qquad (5)$$

where q denotes a quadratfrei number. Note that $3 = 2 + \frac{2}{2}$, $\frac{14}{6} = 2 + \frac{2}{6}$ and $\frac{62}{30} = 2 + \frac{2}{30}$.

Proof. Equation (3) can be written in the following schematic form

$$P_i = i^2 + \sum_{j=1}^{m} d_j i^{g_j} - 2i^{1+\frac{2}{p_h}} + f(i) i^{1+\frac{2}{p_h}} \quad (i \geq 1),$$

where $f(i) \to 0$ and $2 > g_1 > \cdots > g_m > 1 + \frac{2}{p_h}$.

Consequently

$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} i^2 + \sum_{j=1}^{m} d_j \sum_{i=1}^{n} i^{g_j} - 2 \sum_{i=1}^{n} i^{1+\frac{2}{p_h}} + \sum_{i=1}^{n} f(i) i^{1+\frac{2}{p_h}}. \tag{6}$$

If $\alpha = 2, g_1, \ldots, g_m, 1 + \frac{2}{p_h}$ we have

$$\sum_{i=1}^{n} i^\alpha = \int_0^n x^\alpha \, dx + O\left(n^\alpha\right) = \frac{n^{\alpha+1}}{\alpha+1} + O\left(n^\alpha\right) = \frac{n^{\alpha+1}}{\alpha+1} + o\left(n^{2+\frac{2}{p_h}}\right). \tag{7}$$

Note that in equation (7) the left side is a sum of areas of rectangles of basis 1 and height i^α. Therefore, the integral aproximates this sum with error $O\left(n^\alpha\right)$ since x^α is a function strictly increasing.

There exist n_0 such that if $i \geq n_0$ we have $|f(i)| < \epsilon$. Consequently (see (7))

$$\begin{aligned} \left| \sum_{i=1}^{n} f(i) i^{1+\frac{2}{p_h}} \right| &\leq \sum_{i=1}^{n} |f(i)|\, i^{1+\frac{2}{p_h}} \leq \sum_{i=1}^{n_0-1} |f(i)|\, i^{1+\frac{2}{p_h}} + \epsilon \sum_{i=n_0}^{n} i^{1+\frac{2}{p_h}} \\ \leq & \sum_{i=1}^{n_0-1} |f(i)|\, i^{1+\frac{2}{p_h}} + \epsilon \sum_{i=1}^{n} i^{1+\frac{2}{p_h}} = \sum_{i=1}^{n_0-1} |f(i)|\, i^{1+\frac{2}{p_h}} + \epsilon \frac{n^{2+\frac{2}{p_h}}}{2+\frac{2}{p_h}} \\ + & \; o\left(n^{2+\frac{2}{p_h}}\right). \end{aligned}$$

Therefore, from a certain value of n we have

$$\frac{\left| \sum_{i=1}^{n} f(i) i^{1+\frac{2}{p_h}} \right|}{n^{2+\frac{2}{p_h}}} \leq \epsilon.$$

That is

$$\sum_{i=1}^{n} f(i) i^{1+\frac{2}{p_h}} = o\left(n^{2+\frac{2}{p_h}}\right). \tag{8}$$

Finally, substituting equation (7) and equation (8) into (6) we obtain equation (5). The theorem is proved.

Corollary 2.2 *We have* $\sum_{i=1}^{n} P_i \sim \frac{n^3}{3}$.

For example, if $h = 4$ then we have (see (5))

$$\sum_{i=1}^{n} P_i = \frac{1}{3}n^3 - \frac{3}{4}n^{\frac{8}{3}} - \frac{5}{6}n^{\frac{12}{5}} + \frac{13}{7}n^{\frac{7}{3}} - \frac{7}{8}n^{\frac{16}{7}} + o\left(n^{\frac{16}{7}}\right).$$

Now, we prove the following general theorem.

Theorem 2.3 *If A_n is a strictly increasing sequence of positive integers such that*

$$A_n \sim cn^s \qquad (c > 0) \qquad (s > 0) \tag{9}$$

then the following formulae hold

$$\sum_{i=1}^{n} \log A_i = sn \log n - sn + n \log c + o(n), \qquad \prod_{i=1}^{n} A_i = \frac{n^{sn} c^n}{e^{(s+o(1))n}},$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{A_1 \cdots A_n}}{A_n} = \frac{1}{e^s}. \tag{10}$$

Proof. Equation (9) is

$$A_n = f(n)cn^s,$$

where $f(n) \to 1$. Consequently

$$\log A_n = s \log n + \log c + g(n), \tag{11}$$

where $g(n) \to 0$. Therefore

$$\sum_{i=1}^{n} \log A_i = s \sum_{i=1}^{n} \log i + n \log c + \sum_{i=1}^{n} g(i). \tag{12}$$

Now, we have

$$\sum_{i=1}^{n} \log i = \int_1^n \log x + O(\log n) = n \log n - n + O(\log n) = n \log n - n + o(n) \tag{13}$$

On the other hand, there exist n_0 such that if $i \geq n_0$ we have $|g(i)| < \epsilon$. Consequently

$$\left|\sum_{i=1}^{n} g(i)\right| \leq \sum_{i=1}^{n} |g(i)| \leq \sum_{i=1}^{n_0-1} |g(i)| + \epsilon \sum_{n_0}^{n} 1 \leq \sum_{i=1}^{n_0-1} |g(i)| + \epsilon n.$$

Therefore, from a certain value of n we have

$$\frac{\left|\sum_{i=1}^{n} g(i)\right|}{n} < 2\epsilon.$$

That is

$$\sum_{i=1}^{n} g(i) = o(n). \tag{14}$$

Equations (12), (13) and (14) give

$$\sum_{i=1}^{n} \log A_i = sn \log n - sn + n \log c + o(n). \tag{15}$$

Equation (15) gives

$$\prod_{i=1}^{n} A_i = \exp\left(\sum_{i=1}^{n} \log A_i\right) = \frac{n^{sn} c^n}{e^{(s+o(1))n}}$$

Equations (11) and (15) give

$$\log \frac{\sqrt[n]{A_1 \cdots A_n}}{A_n} = \frac{\sum_{i=1}^{n} \log A_i}{n} - \log A_n = -s + o(1). \tag{16}$$

Equation (10) is an immediate consequence of equation (16). The theorem is proved.

If $A_n = P_n$ then equation (1) give us the following corollary.

Corollary 2.4 *The following formulae hold*

$$\sum_{i=1}^{n} \log P_i = 2n \log n - 2n + o(n), \qquad \prod_{i=1}^{n} P_i = \frac{n^{2n}}{e^{(2+o(1))n}},$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{P_1 \cdots P_n}}{P_n} = \frac{1}{e^2}.$$

Theorem 2.5 *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{1}{\sqrt{P_i}} = \log n + C + o(1), \tag{17}$$

where C is a constant.

Proof. We have (see the proof of Lemma 5 in [1])

$$\sqrt{P_n} = n - f(n) n^{\frac{2}{3}} \qquad (n \geq 1),$$

where $f(n) \to 1$. Therefore

$$\frac{1}{\sqrt{P_n}} = \frac{1}{n - f(n) n^{\frac{2}{3}}} = \frac{1}{n}\left(\frac{1}{1 - f(n) n^{-\frac{1}{3}}}\right) \qquad (n \geq 1). \tag{18}$$

We have the following equation

$$\frac{1}{1-x} = 1 + \frac{1}{1-x}x = 1 + g(x)x \qquad (x \neq 1), \tag{19}$$

where

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{1-x} = 1.$$

Equations (18) and (19) give us

$$\begin{aligned} \frac{1}{\sqrt{P_n}} &= \frac{1}{n}\left(\frac{1}{1-f(n)n^{-\frac{1}{3}}}\right) = \frac{1}{n}\left(1 + g\left(f(n)n^{-\frac{1}{3}}\right) f(n)n^{-\frac{1}{3}}\right) \\ &= \frac{1}{n}\left(1 + h(n)n^{-\frac{1}{3}}\right) = \frac{1}{n} + h(n)\frac{1}{n^{\frac{4}{3}}} \qquad (h(n) \to 1) \quad (n \geq 1) \end{aligned} \tag{20}$$

It is well-known the equation (see [2], page 95)

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + o(1), \tag{21}$$

where γ is called Euler's constant.

Equations (20) and (21) give

$$\sum_{i=1}^{n} \frac{1}{\sqrt{P_i}} = \sum_{i=1}^{n} \frac{1}{i} + \sum_{i=1}^{n} h(i)\frac{1}{i^{\frac{4}{3}}} = \log n + \gamma + o(1) + \sum_{i=1}^{n} h(i)\frac{1}{i^{\frac{4}{3}}}. \tag{22}$$

Note that from a certain value of i we have $0 < h(i) < h$ $(h > 1)$. Consequently $\sum h(i)\frac{1}{i^{\frac{4}{3}}} < h \sum \frac{1}{i^{\frac{4}{3}}}$. Now, the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{4}{3}}}$ converges, since $4/3 > 1$. Therefore the series (comparison test)$\sum_{i=1}^{\infty} h(i)\frac{1}{i^{\frac{4}{3}}}$ also converges. That is $\sum_{i=1}^{\infty} h(i)\frac{1}{i^{\frac{4}{3}}} = E$. Therefore

$$\sum_{i=1}^{n} h(i)\frac{1}{i^{\frac{4}{3}}} = E + o(1). \tag{23}$$

Equations (22) and (23) give (17) with $C = \gamma + E$. The theorem is proved.

Theorem 2.6 *The following asymptotic formula holds*

$$\sum_{P_n \leq x} \frac{1}{\sqrt{P_n}} = \frac{1}{2}\log x + C + o(1). \tag{24}$$

Proof. We have (see (17) and (11))

$$\sum_{P_i \leq P_n} \frac{1}{\sqrt{P_i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{P_i}} = \log n + C + o(1) = \frac{1}{2}\log P_n + C + f(n), \tag{25}$$

where $f(n) \to 0$. If we write $g(P_n) = f(n)$ then equation (25) becomes

$$\sum_{P_i \le P_n} \frac{1}{\sqrt{P_i}} = \frac{1}{2}\log P_n + C + g(P_n). \tag{26}$$

If $P_n \le x < P_{n+1}$ let us consider the function $g(x) = g(P_n)$. We have $\lim_{x\to\infty} g(x) = 0$. That is,

$$g(x) = o(1). \tag{27}$$

If $P_n \le x < P_{n+1}$ let us consider the function $\frac{1}{2}\log x - \frac{1}{2}\log P_n$. Since $\frac{1}{2}\log P_n \le \frac{1}{2}\log x < \frac{1}{2}\log P_{n+1}$ we obtain

$$0 \le \frac{1}{2}\log x - \frac{1}{2}\log P_n \le \frac{1}{2}\log P_{n+1} - \frac{1}{2}\log P_n. \tag{28}$$

We have (see (11)) $\log P_n = 2\log n + h(n)$, where $h(n) \to 0$. Therefore (mean value Theorem)

$$\begin{aligned} &\frac{1}{2}\log P_{n+1} - \frac{1}{2}\log P_n = \frac{1}{2}\left(2\log(n+1) + h(n+1) - 2\log n - h(n)\right) \\ = \ &(\log(n+1) - \log n) + \frac{1}{2}h(n+1) - \frac{1}{2}h(n) = \frac{1}{n+\epsilon(n)} + \frac{1}{2}h(n+1) \\ - \ &\frac{1}{2}h(n) \to 0 \qquad (0 < \epsilon(n) < 1). \end{aligned} \tag{29}$$

Equations (29) and (28) give $\lim_{x\to\infty}\left(\frac{1}{2}\log x - \frac{1}{2}\log P_n\right) = 0$. That is

$$\frac{1}{2}\log x - \frac{1}{2}\log P_n = o(1). \tag{30}$$

If $P_n \le x < P_{n+1}$ we have (see (26))

$$\begin{aligned} \sum_{P_i \le x} \frac{1}{\sqrt{P_i}} &= \sum_{P_i \le P_n} \frac{1}{\sqrt{P_i}} = \frac{1}{2}\log P_n + C + g(P_n) \\ &= \frac{1}{2}\log x - \left(\frac{1}{2}\log x - \frac{1}{2}\log P_n\right) + C + g(x). \end{aligned} \tag{31}$$

Equation (31), equation (27) and equation (30) give equation (24). The theorem is proved.

References

[1] R. Jakimczuk, Asymptotic formulae for the n-th perfect power, *J. Integer Seq.* **15** (2012), Article 12.5.5.

[2] W. J. LeVeque, *Topics in Number Theory,* Addison-Wesley, 1958.

Received: October, 2012