Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 4, 163 - 171
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Track Properities of Normal Chain

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Abstract

In this paper, research state space structure properities of normal chain from Ray-Knight theory on a given set E: If $S_{\infty}(\omega)$ whose probability is 1 are empty set,then,the normal chain only has jumping type track, at the time, the transfer function must be the least and only determined by desity matrix; If $S_{\infty}(\omega)$ whose probability is positive are not empty set, then,the track of normal chain is not jumping type track, and its transfer function cannot determine by density matrix.

Keywords: Normal chain; Stopping time, Jumping track

1 Introduction

In the classic books of some Markov chain[1−3],for the transfer function in a given E, the usuall practice is constructing canonical chain on $E \bigcup \infty$. The state space of model chain is very simply. But its track just has right lower semi continuity, and just keep part of the strong Markov. As application of Ray-night theory, the normal chain can overcome these defects[4−5],but the structure of state space may be very complex. So we will research the properities of the normal chain.

2 Preliminary knowledge[3]

Let $E=\{1,2,\cdots\}$, $P(t)=(p_{ij}(t))_{i,j\in E}$, and $P(t)$ is the honest transfer function on E, $Q=(q_{ij})_{i,j\in E}$, Q is density matrix of $P(t)$, $R_{ij}(\lambda)$ is a resolvent of $P(t)$, $\overline{E}$ is Ray-Knight tight for $P(t)$ on $E,(U^{\alpha})_{\alpha>0}$ is a Ray resolvent for $P(t)$. And $P(t)_{t\geq 0}$ is Ray semi group for $P(t)$, ε is a Borel algebra on E^{+}.Let $D_{\overline{E}}[0,\infty)=\{\omega \mid \omega:[0,\infty)\to\overline{E}\}$, F is a Borel algebra on $D_{\overline{E}}[0,\infty)$, Coordinate process of F is $\{X_t\}$, Let Ω denote the subset $\{T^{-}_{E_R}=\infty, T_{(E^{+})^{c}}=\infty\}$ of $D_{\overline{E}}[0,\infty)$. θ_t is the mapping: $\theta_t:\omega(\cdot)\mapsto\omega(t+\cdot),\omega(\cdot)\in D_{\overline{E}}[0,\infty),\forall t.P^{x}$ is the probability measure of x on E^{+}.

Definition 1[6] Let $X=(\Omega,F,F_t,X_t,\theta_t,P^{x})$, X is called **normal chain** on $P(t)$.

Definition 2 For a funcation T on $\{\Omega,F\},T:\Omega\to[0,\infty)$, if for any $t\geq 0$, there are $\{T\leq t\}\in F_t$, then we called T is **stopping time**.

Definition 3 Let $S_i(\omega)=\{s \mid X_s(\omega)=i\}$, we call $S_i(\omega)$ is **i-constant value set** ; the interval of $S_i(\omega)$ is called **i-interval of** X .

Definition 4 If $T_{fl} = \inf\{s \mid s \ge 0, X_s \ne X_0\}$, $T_{re} = \inf\{s \mid s > 0, X_s = X_0\}$, T_{fl} is called **escape time**. T_{re} is called **return time.**

Definition 5 For any $x \in E^+, P^x\{T_{fl} = 0\} = 0$ or 1, $P^x\{T_{re} = 0\} = 0$ or 1. If $P^x\{T_{fl} = \infty\} = 1$, x is called **absorbing state**; If $P^x\{T_{fl} > 0\} = 1$, x is called **stay state**; If $P^x\{T_{re} = 0\} = 1$, x is called **regular state**; If $P^x\{T_{re} = 0\} = P^x\{T_{fl} = 0\} = 1, x$ is called **instantaneous state**.

3 Relevant theorems

Theorem 1 Let $i \in E$, then

(1) i is Regular State.

(2) On P^i, T_{fl} obey exponential distribution of parameter q_i.

(3) On P^i, $X_{T_{fl}}$ and T are independent.

(4) If $0 < q_i < \infty$, then for any $j \in E, j \ne i, P^i\{X_{T_{fl}} = j\} = \dfrac{q_{ij}}{q_i}$.

Proof: (1) If i is not Regular State, then $P^i\{T_{re} > 0\} = 1$, for any $t > 0$, it is easy to see $\{X_t = i\} \subseteq \{T_{re} \le t\}$ and when $t \to 0$, $\{T_{re} \le t\} \to \{T_{re} = 0\}$, so we have that

$$1 = \lim_{t\to 0} P_{ii}(t) = \lim_{t\to 0} P^i\{X_t = i\} \le \lim_{t\to 0} P^i\{T_{re} \le t\} = 0$$

This is a contradiction result. So i is Regular State.

(2) Refer to [2].

(3) If $q_i = 0$ or ∞, it is clearly that $P^i\{T_{fl} = \infty\} = 1$ or $P^i\{T_{fl} = 0\} = 1$.

If $0 < q_i < \infty$,for any $A, A \subseteq E^+$, and $t, s > 0$,

$$P^i\{T_{fl} > t + s, X_{T_{fl}} \in A\} = P^i\{T_{fl} > t, T_{fl} \circ \theta_t > s, X_{T_{fl}} \circ \theta_t \in A\}$$

$$= E^i\{P^i\{T_{fl} \circ \theta_t > s, X_{T_{fl}} \circ \theta_t \in A \mid F_t\}; T_{fl} > t\}$$

$$= E^i\{P^{X_t}\{T_{fl} > s, X_{T_{fl}} \in A\}; T_{fl} > t\}$$

$$= P^i\{T_{fl} > t\} P^i\{T_{fl} > s, X_{T_{fl}} \in A\}$$

Let $s \to 0$, then $P^i\{T_{fl} > t, X_{T_{fl}} \in A\} = P^i\{T_{fl} > t\} P^i\{X_{T_{fl}} \in A\}$

So, on P^i, T_{fl} and $X_{T_{fl}}$ are independent.

(4) If $0 < q_i < \infty$, for any $j \neq i$, $\lambda > 0$, from (3) and the strong Markov of X ,

We get that

$$R_{ij}(\lambda) = \int_0^\infty e^{-\lambda} p_{ij}(t) dt = E^i[\int_0^\infty e^{-\lambda t} I_{\{j\}}(X_t) dt]$$

$$= E^i[\int_{T_{fl}}^\infty e^{-\lambda t} I_{\{j\}}(X_t) dt]$$

$$= E^i[e^{-\lambda T_{fl}}[\int_0^\infty e^{-\lambda t} I_{\{j\}}(X_t) dt] \circ \theta_{T_{fl}}]$$

$$= E^i[e^{-\lambda T_{fl}} E^{X_{T_{fl}}}[\int_0^\infty e^{-\lambda t} I_{\{j\}}(X_t) dt]]$$

$$= E^i[e^{-\lambda T_{fl}} \int_0^\infty e^{-\lambda t} P^{X_{T_{fl}}}[X_t = j] dt]$$

$$= E^i[e^{-\lambda T_{fl}} \int_0^\infty e^{-\lambda t} P_t(X_{T_{fl}}, \{j\}) dt]$$

$$= E^i[e^{-\lambda T_{fl}} U^\lambda(X_{T_{fl}}, \{j\})]$$

$$= E^i[e^{-\lambda T_{fl}}] E^i[U^\lambda(X_{T_{fl}}, \{j\})]$$

For any $x \in E^{+}, x \neq j$, for continuous functions $f(\cdot)$ on E, so that $f(x) = 0,\ f(j) = 1$, then

$$0 = f(x) = \lim_{\lambda \to \infty} \lambda U^{\lambda} f(x) \geq \lim_{\lambda \to \infty} \lambda U^{\lambda}(x, \{j\}).$$

But $\lim_{\lambda \to \infty} \lambda U^{\lambda}(j, \{j\}) = \lim_{\lambda \to \infty} \lambda R_{jj}(\lambda) = 1$,

So $q_{ij} = \lim_{\lambda \to \infty} \lambda^{2} R_{ij}(\lambda) = \lim_{\lambda \to \infty} \lambda E^{i}[e^{-\lambda T_{\beta}}] \lim_{\lambda \to \infty} \lambda E^{i}[U\lambda(X_{T_{\beta}}, \{j\})]$

$$= \lim_{\lambda \to \infty} \frac{\lambda q_{i}}{\lambda + q_{i}} \cdot E^{i}[\lim_{\lambda \to \infty} U^{\lambda}(X_{T_{\beta}}, \{j\})]$$

$$= q_{i} \cdot P^{i}[X_{T_{\beta}} = j]$$

That is $P^{i}[X_{T_{\beta}} = j] = \dfrac{q_{ij}}{q_{i}}$.

Note1: From Theorem 1 we get that $i \in E$ is instantaneous state(stable state) if and only is that i is instantaneous state(stay state) of normal chain X.

Theorem 2 If $q_{i} < \infty$, then for any $x \in E^{+}$, there are $a_{1}, b_{1}, a_{2}, b_{2} \cdots$, so that $a_{k} \leq b_{k}$ and if $a_{k} < \infty$, then $a_{k} < b_{k}$; If $b_{k} < \infty, b_{k} < a_{k+1}$, for any k, will have $S_{i}(\omega) = \bigcup_{k} [a_{k}(\omega), b_{k}(\omega))$. For any $s < t$, let $\xi_{i}(s,t)$ denote the number of $[a_{k}, b_{k})$ in $[s,t]$, $k = 1, 2, \cdots$, then $E^{x}\{\xi_{i}(s,t)\} \leq q_{i}(t-s)$.

Proof: Let

$$a_1 = \inf\{u \mid u \ge 0, X_u = i\}$$
$$b_1 = \inf\{u \mid u \ge a_1, X_u \ne i\}$$
$$a_{k+1} = \inf\{u \mid u \ge b_k, X_u = i\}$$
$$b_{k+1} = \inf\{u \mid u \ge a_k, X_u \ne i\}$$

It is easy to know that $a_k, b_k, k = 1, 2, \cdots$ are $\{F_t\}-$stopping time. For any k , if $a_k < \infty$, because X is right continuous , and $X_{a_k} = i$, from (2) of Theorem 1,we have that

$$P^x[a_k < \infty, b_k > a_k] = P[a_k < \infty, T_{fl} \circ \theta_{a_k} > 0]$$
$$= E^x[P^x[T_{fl} \circ \theta_{a_k} > 0 \mid F_{a_k}]; a_k < \infty]$$
$$= E^x[P^i[T_{fl} > 0]; a_k < \infty]$$
$$= P^x[a_k < \infty]$$

Then , on $\{a_k < \infty\}$, almost sure have $a_k < b_k$, from the strong Markov of X and define of b_k, for any k, we get that

$$0 = P^x[b_k < \infty, \exists \varepsilon > 0, X \in [b_k, b_k + \varepsilon), X = i]$$
$$= P^x\{b_k < \infty, X_{b_k} = i, T_{fl} \circ \theta_{b_k} > 0\}$$
$$= E^x[P^x[T_{fl} \circ \theta_{b_k} > 0 \mid F_{b_k}]; b_k < \infty, X_{b_k} = i]$$
$$= E^x[P^i[T_{fl} > 0]; b_k < \infty, X_{b_k} = i]$$
$$= P^x\{X_{b_k} = i, b_k < \infty\}$$

So , on $\{b_k < \infty\}$, almost sure have $X_{b_k} \ne i$.For any $0 < s < t$, it is obvious that $\xi_i(s,t)\} = \xi_i(0, t-s) \circ \theta_s$.(refer to [2](120-130))

$$E^x\{\xi_i(s,t)\} \le E^x\{\xi_i(0,t-s)\circ\theta_s\}$$
$$= \sum P^x\{X_s = k\}E^x\{\xi_i(0,t-s)\}$$
$$\le q_i \cdot (t-s)$$

For any $t > 0$, have $E^x\{\xi_i(0,t)\} \le q_i \cdot t$ .So ,almost sure that there are only limited $[a_k, b_k), k = 1, 2, \cdots$ on any limited interval. Then $\lim_{k\to\infty} a_k = \infty$, we may get that

$$S_i(\omega) = \bigcup_k [a_k(\omega), b_k(\omega))$$

Theorem 3 If $q_i = \infty$, we have:

(1) Almost sure $S_i(\omega)$ does not contain any interval.

(2) Almost sure $S_i(\omega)$ is own dense set.

Proof:(1) It is clear that $S_i(\omega)$ is a optional set. Let

$$A_t(\omega) = \sup\{s \,|\, s < t, s \notin S_i(\omega)\}, t \ge 0, \omega \in \Omega\},$$

it is obvious $\{A_t\}$ is monotone increasing light continuous process, and it suitable $\{F_t\}$. Let $B_t = \lim_{s\to t} A_s$, then $\{B_t\}$ is a right continuous process of optional suitable $\{F_t\}$.Set $U = \{(\omega,t) \,|\, \exists\varepsilon > 0, (t-\varepsilon, t+\varepsilon) \subseteq S_i(\omega)\}, \omega \in \Omega$, then $U = \{(\omega,t) \,|\, B_t(\omega) < t\}$, so U is the optional set suitable $\{F_t\}$.

We use D_u to denote the opportunity time U, If $P^x\{D_U < \infty\} > 0$, there exists $\{F_t\}-$ stopping time T , so that $P^x\{T < \infty\} > 0$, and on $\{T < \infty\}, (\omega, T(\omega)) \in U$, From (2) of theorem 1 we have that

$$P^x[T < \infty] = P^x[T < \infty, X_T = i, \exists\varepsilon > 0, \varepsilon \in (T-\varepsilon, T+\varepsilon), X \equiv i]$$

$$\leq P^x[T < \infty, X_T = i, T_{fl} \circ \theta_T > 0]$$
$$= E^x[P^i[T_{fl} > 0], T < \infty] = 0$$

This is a contradiction result! So $P^x\{D_U < \infty\} = 0$ .Namely, almost sure $S_i(\omega)$ does not contain any interval.

(2) Proof methods is same to (1).

4 Main Results

Proposition1 If $x \in E^+ \setminus E$,then $P^x\{T_{fl} = 0\} = 1$.

Proof: For any $i \in E, q_i < \infty$, we use $[a_k^{(i)}, b_k^{(i)}]$ denote the k-th i-interval of $S_i(\omega)$, Let $S_f(\omega) = \{u \mid X_U(\omega) \notin E\}$, S_f is the time of stopping the virtualstate. For any $x \in E^+, t > 0$,

$$E^x\{S_f \cap [0,t]\} = E^x\{\int_0^t [1 - I_E(X_s)]ds\}$$
$$= \int_0^t [1 - P^x\{X_s \in E\}]ds = 0$$

Then , meas S_f =0, if $x \in E^+ \setminus E$, we have that $P^x\{T_{fl} = 0\} = 1$.

Proposition2 If $\Omega = (q_{ij})_{i,j \in E}$ is all stability, For any $\omega \in \Omega$, Let $S_\infty(\omega) = \{s \mid s \geq 0, \forall \varepsilon > 0;$ it will have infinite jumps$\}$, then $S_\infty(\omega)$ is closed set. $S_\infty(\omega) \setminus S_f(\omega)$ contain countable points, and meas $S_i(\omega)$ almost sure is 0.

Proposition3 Let $\sigma(\omega) = \inf\{s \mid s > 0, s \in S_\infty(\omega)\}, \omega \in \Omega$, then σ is $\{F_t\}-$Stopping time.

Proposition4 The campaign on $(0,\infty) \setminus S_\infty(\omega)$ is determined by the minimum transfer function.

Proof: For any n and $t \geq 0$, $i, j \in E$, we denote:

$$h_{ij}^{(n)}(t) = P[X_t = j.[0,t] \text{ have } n \text{ jumps}]$$

It is easy that:

$$h_{ij}^{(0)}(t) = e^{-q_i^t}\delta_{ij}, \quad h_{ij}^{(n)}(t) = \sum_{k \neq i} \int_0^t e^{-q_{is}} q_{ik} h_{kj}^{n-1}(t-s)ds, n = 1, 2, \cdots$$

$$P^i[X_t = j, t < \sigma] = \sum h_n^{(n)}(t) - p_{ij}^{\min}(t), \forall i, j \in E, t \geq 0$$

Because $(0,\infty) \setminus S_\infty(\omega)$ is a open set, and all track on the $(0,\infty) \setminus S_\infty(\omega)$ are jumping type, for any $\{F_t\}-$stopping time T,

$$P^\mu[X_{T+t} = j, [T, T+t] \cap S_\infty(\omega) = \varnothing \big| X_T = i] = P_{ij}^{\min}(t)$$

So the campaign on $(0,\infty) \setminus S_\infty(\omega)$ is determined by the minimum transfer function.

Note2: If $S_\infty(\omega)$ whose probability is 1 is empty set, then, the normal chain only has jumping type track, at the time, the transfer function must be the least and be only determined by density matrix. If $S_\infty(\omega)$ whose probability is positive is not empty set, then the track of normal chain is not jumping type track, and its transfer function cannot determine by density matrix.

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Received: November, 2012