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On the Quaternionic Smarandache Curves in Euclidean 3-Space

Muhammed Çetin

Celal Bayar University
Instutition of Science and Technology,Manisa, Turkey
mat.mcetin@hotmail.com

Hüseyin Kocayiğit

Celal Bayar University
Faculty of Arts and Science
Department of Mathematics
Muradiye Campus, 45140 Manisa, Turkey
huseyin.kocayigit@bayar.edu.tr

Abstract

In this paper, we give the definitions of quaternionic Smarandache curves in 3-dimensional Euclidean space E^3 and we investigate some differential geometric properties of these curves.

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1. Introduction

The quaternion was introduced by Hamilton. His initial attempt to generalise the complex numbers by introducing a 3-dimensional object failed in the sense that the algebra he constructed for these 3-dimensional objects did not have the desired properties. On 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar (real) axis is left unchanged whereas the vector (imaginary) axis is supplemented by adding two further vector axes [11].

Some differential geometers have studied on special curves by using the quaternions. Güngör and Tosun have studied the spatial quaternionic rectifying curves in Euclidean space R^3 [7]. In [9,10] Önder has given the definitions and characterizations of spatial quaternionic Salkowski, anti-Salkowski and similar curves and he has introduced the quaternionic similar curves 4-dimensional Euclidean space E^4. Gök, Okuyucu, Kahraman and Hacısalihoğlu have given a new defnition of harmonic curvature functions in terms of B_2 and they have defined a new kind of slant helix which they have called B_2–slant helix in 4-dimensional Euclidean space E^4 by using the new harmonic curvature functions [5]. Kahraman, Gök and Hacısalihoğlu have studied quaternionic slant helices which they have called semi-real quaternionic B_2-slant helix in four dimensional semi-Euclidean space E_2^4 [3]. Çöken and Tuna have studied quaternion valued functions and quaternionic inclined curves in the semi-Euclidean space E_2^4 [1]. They have given the Serret-Frenet formulae for the quaternionic curve in the semi-Euclidean space E_2^4. Then they have defined quaternionic inclined curves and harmonic curvatures for the quaternionic curves in the semi-Euclidean space E_2^4.

In [2] Ali has introduced some special Smarandache curves in the Euclidean space. He has studied Frenet-Serret invariants of a special case. Çetin, Tunçer and Karacan have investigated special Smarandache curves according to Bishop frame in Euclidean 3-space. Furthermore, they have found differential geometric properties of these special curves and they have calculated first and second curvature (natural curvatures) of these curves. Also they have defined the centers of the curvature spheres and osculating spheres of Smarandache curves [8].

In this study, we define spatial quaternionic Smarandache curves in 3-dimensional Euclidean space E^3 and we give some differential geometric properties of these curves.

2. Preliminaries

In this section we give the basic elements of the quaternions and quaternionic curves. More details can be investigated in [6,4,11].

A real quaternion is defined by $q = d + ae_1 + be_2 + ce_3$ where a, b, c and d are real numbers. This equation can be written as $q = S_q + V_q$ where $S_q = d$ is the scalar part and $V_q = ae_1 + be_2 + ce_3$ is the vector part of the q.

The conjugate of the quaternion $q = S_q + V_q$ is defined by

$$\alpha q = S_q - V_q = d - ae_1 - be_2 - ce_3 .$$

q is called a spatial quaternion if $q + \alpha q = 0$. This means that $q = V_q$. And q is called a temporal quaternion if $q - \alpha q = 0$.

The algebra of the quaternions denoted by Q. Then, the quaternion inner product is defined by

$$h : Q \times Q \to R, \quad (p,q) \to h(p,q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p).$$

Here, h is the symmetric, real-valued, non-degenerate and bilinear form.

The quaternion product of p and q is defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p + V_p \wedge V_q - \langle V_p, V_q \rangle.$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product in E^3, respectively.

The norm of the quaternion q is given by

$$\|q\|^2 = h(q,q) = q \times \alpha q = \alpha q \times q = a^2 + b^2 + c^2 + d^2.$$

If $\|q\| = 1$, then q is called a unit quaternion.

The inverse of the quaternion q is given by

$$q^{-1} = \frac{\alpha q}{\|q\|}.$$

Theorem 1. The three-dimensional Euclidean space E^3 is identified with the space of spatial quaternions $\{q \in Q : q + \alpha q = 0\}$ in an obvious manner. Let $I = [0,1]$ be an interval in real line R and let

$$\alpha : I \subset R \to Q, \quad s \to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i$$

be an arc-lengthed curve with nonzero curvatures $\{k_\alpha, r_\alpha\}$ and $\{t_\alpha(s), n_1^\alpha(s), n_2^\alpha(s)\}$ denotes the Frenet frame of the curve $\alpha(s)$, where $t_\alpha(s)$ is unit tangent, $n_1^\alpha(s)$ is unit principal normal and $n_2^\alpha(s)$ is unit binormal of the curve $\alpha(s)$. Then Frenet formulae of the quaternionic curve $\alpha(s)$ are given by

$$\begin{bmatrix} t_\alpha' \\ (n_1^\alpha)' \\ (n_2^\alpha)' \end{bmatrix} = \begin{bmatrix} 0 & k_\alpha & 0 \\ -k_\alpha & 0 & r_\alpha \\ 0 & -r_\alpha & 0 \end{bmatrix} \begin{bmatrix} t_\alpha \\ n_1^\alpha \\ n_2^\alpha \end{bmatrix}$$

where k_α is the principal curvature and r_α is the torsion of the quaternionic curve $\alpha(s)$ [6].

3. On the Quaternionic Smarandache Curves in E^3

In [2], author gave following definitions:

Definition 1. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache TN curves are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T + N).$$

Definition 2. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache NB curves are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(N + B).$$

Definition 3. Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Serret-Frenet frame. Smarandache TNB curves are defined by

$$\beta(s^*) = \frac{1}{\sqrt{3}}(T + N + B).$$

In this section, we investigate spatial quaternionic Smarandache curves in R^3.

3.1. Quaternionic tn_1-Smarandache Curves.

Definition 4. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternionic curve in E^3 and $\{t_\alpha, n_1^\alpha, n_2^\alpha\}$ be its moving Serret-Frenet frame. Quaternionic tn_1-Smarandache curves can be defined by

$$\gamma(s^*) = \frac{1}{\sqrt{2}}(t_\alpha + n_1^\alpha) \tag{3.1}$$

Now, we can investigate Serret-Frenet invariants of quaternionic tn_1-Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.1) with respect to s, we get

$$\dot{\gamma} = \frac{d\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-k_\alpha t_\alpha + k_\alpha n_1^\alpha + r_\alpha n_2^\alpha) \tag{3.2}$$

and

$$t_\gamma \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-k_\alpha t_\alpha + k_\alpha n_1^\alpha + r_\alpha n_2^\alpha)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_\alpha^2 + r_\alpha^2}{2}}. \tag{3.3}$$

The tangent vector of curve γ can be written as follow,

$$t_\gamma = \frac{1}{\sqrt{2k_\alpha^2 + r_\alpha^2}}(-k_\alpha t_\alpha + k_\alpha n_1^\alpha + r_\alpha n_2^\alpha). \tag{3.4}$$

Differentiating (3.4) with respect to s, we obtain

$$\frac{dt_\gamma}{ds^*}\frac{ds^*}{ds}=\frac{1}{(2k_\alpha^2+r_\alpha^2)^{\frac{3}{2}}}(\lambda_1 t_\alpha+\lambda_2 n_1^\alpha+\lambda_3 n_2^\alpha) \tag{3.5}$$

where

$$\lambda_1=-2k_\alpha^4-\dot{k}_\alpha r_\alpha^2-k_\alpha^2 r_\alpha^2+k_\alpha r_\alpha \dot{r}_\alpha$$
$$\lambda_2=-2k_\alpha^4-3k_\alpha^2 r_\alpha^2+\dot{k}_\alpha r_\alpha^2-r_\alpha^4-k_\alpha r_\alpha \dot{r}_\alpha$$
$$\lambda_3=2k_\alpha^3 r_\alpha+2k_\alpha^2 \dot{r}_\alpha+k_\alpha r_\alpha^3-2k_\alpha \dot{k}_\alpha r_\alpha.$$

Substituting (3.3) in (3.5), we get

$$t_\gamma{}'=\frac{\sqrt{2}}{(2k_\alpha^2+r_\alpha^2)^2}(\lambda_1 t_\alpha+\lambda_2 n_1^\alpha+\lambda_3 n_2^\alpha).$$

Then, the principal curvature and principal normal vector field of curve γ are respectively,

$$k_\gamma=\|t_\gamma{}'\|=\frac{\sqrt{2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{(2k_\alpha^2+r_\alpha^2)^2}$$

and

$$n_1^\gamma=\frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}(\lambda_1 t_\alpha+\lambda_2 n_1^\alpha+\lambda_3 n_2^\alpha).$$

On the other hand, we express

$$n_2^\gamma=-\left\langle V_{t_\gamma},V_{n_1^\gamma}\right\rangle+V_{t_\gamma}\wedge V_{n_1^\gamma}.$$

So, the binormal vector of curve γ is

$$n_2^\gamma=\frac{1}{\sqrt{2k_\alpha^2+r_\alpha^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}(\sigma_4+\sigma_1 t_\alpha+\sigma_2 n_1^\alpha+\sigma_3 n_2^\alpha)$$

where

$$\sigma_1=k_\alpha\lambda_3-r_\alpha\lambda_2$$
$$\sigma_2=k_\alpha\lambda_3+r_\alpha\lambda_1$$
$$\sigma_3=-k_\alpha(\lambda_1+\lambda_2)$$
$$\sigma_4=k_\alpha\lambda_1-k_\alpha\lambda_2-r_\alpha\lambda_3.$$

We differentiate (3.2) with respect to s in order to calculate the torsion

$$\ddot{\gamma}=\frac{1}{\sqrt{2}}\left[(-\dot{k}_\alpha-k_\alpha^2)t_\alpha+(-k_\alpha^2+\dot{k}_\alpha-r_\alpha^2)n_1^\alpha+(k_\alpha r_\alpha+\dot{r}_\alpha)n_2^\alpha\right]$$

and similarly

$$\ddot{\gamma}=\frac{1}{\sqrt{2}}(\rho_1 t_\alpha+\rho_2 n_1^\alpha+\rho_3 n_2^\alpha)$$

where

$$\rho_1 = -\ddot{k}_\alpha - 3k_\alpha \dot{k}_\alpha + k_\alpha^3 + k_\alpha r_\alpha^2$$
$$\rho_2 = -3k_\alpha \dot{k}_\alpha - k_\alpha^3 - 3r_\alpha \dot{r}_\alpha + \ddot{k}_\alpha - k_\alpha r_\alpha^2$$
$$\rho_3 = -k_\alpha^2 r_\alpha + 2\dot{k}_\alpha r_\alpha - r_\alpha^3 + k_\alpha \dot{r}_\alpha + \ddot{r}_\alpha .$$

The torsion of curve γ is

$$r_\gamma = \frac{\sqrt{2}\begin{bmatrix} (\dot{k}_\alpha + k_\alpha^2)(k_\alpha\rho_3 - r_\alpha\rho_2) + (k_\alpha^2 - \dot{k}_\alpha + r_\alpha^2)(k_\alpha\rho_3 + r_\alpha\rho_1) \\ + k_\alpha(k_\alpha r_\alpha + \dot{r}_\alpha)(\rho_1 + \rho_2) \end{bmatrix}}{\begin{bmatrix} (2k_\alpha \dot{k}_\alpha + r_\alpha \dot{r}_\alpha)^2 + (2k_\alpha^2 r_\alpha + k_\alpha \dot{r}_\alpha - \dot{k}_\alpha r_\alpha + r_\alpha^3)^2 \\ + (k_\alpha \dot{r}_\alpha - \dot{k}_\alpha r_\alpha)^2 + (2k_\alpha^3 + k_\alpha r_\alpha^2)^2 \end{bmatrix}} .$$

3.2. Quaternionic tn_2 -Smarandache Curves.

Definition 5. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternionic curve in E^3 and $\{t_\alpha, n_1^\alpha, n_2^\alpha\}$ be its moving Serret-Frenet frame. Quaternionic tn_2 -Smarandache curves can be defined by

$$\gamma(s^*) = \frac{1}{\sqrt{2}}(t_\alpha + n_2^\alpha) \tag{3.6}$$

Now, we can investigate Serret-Frenet invariants of quaternionic tn_2 Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.6) with respect to s, we get

$$\dot{\gamma} = \frac{d\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(k_\alpha - r_\alpha)n_1^\alpha \tag{3.7}$$

and

$$t_\gamma \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(k_\alpha - r_\alpha)n_1^\alpha$$

where

$$\frac{ds^*}{ds} = \frac{k_\alpha - r_\alpha}{\sqrt{2}} . \tag{3.8}$$

The tangent vector of curve γ can be written as follow,

$$t_\gamma = n_1^\alpha . \tag{3.9}$$

Differentiating (3.9) with respect to s, we obtain

$$\frac{dt_\gamma}{ds^*}\frac{ds^*}{ds} = -k_\alpha t_\alpha + r_\alpha n_2^\alpha \tag{3.10}$$

Substituting (3.8) in (3.10), we get

$$t_\gamma{}'=\frac{\sqrt{2}}{k_\alpha - r_\alpha}(-k_\alpha t_\alpha + r_\alpha n_2^\alpha)\,.$$

Then, the principal curvature and principal normal vector field of curve γ are respectively,

$$k_\gamma = \left\| t_\gamma{}' \right\| = \frac{\sqrt{2(k_\alpha^2 + r_\alpha^2)}}{k_\alpha - r_\alpha}$$

and

$$n_1^\gamma = \frac{1}{\sqrt{k_\alpha^2 + r_\alpha^2}}(-k_\alpha t_\alpha + r_\alpha n_2^\alpha)\,.$$

On the other hand, we express

$$n_2^\gamma = -\left\langle V_{t_\gamma}, V_{n_1^\gamma} \right\rangle + V_{t_\gamma} \wedge V_{n_1^\gamma}\,.$$

So, the binormal vector of curve γ is

$$n_2^\gamma = \frac{1}{\sqrt{k_\alpha^2 + r_\alpha^2}}(r_\alpha t_\alpha + k_\alpha n_2^\alpha)\,.$$

We differentiate (3.7) with respect to s in order to calculate the torsion

$$\ddot{\gamma} = \frac{1}{\sqrt{2}}\left[(-k_\alpha^2 + k_\alpha r_\alpha) t_\alpha + (\dot{k}_\alpha - \dot{r}_\alpha) n_1^\alpha + (k_\alpha r_\alpha - r_\alpha^2) n_2^\alpha \right]$$

and similarly

$$\ddot{\gamma} = \frac{1}{\sqrt{2}}(\rho_1 t_\alpha + \rho_2 n_1^\alpha + \rho_3 n_2^\alpha)$$

where

$$\rho_1 = -3k_\alpha \dot{k}_\alpha + \dot{k}_\alpha r_\alpha + 2k_\alpha \dot{r}_\alpha$$

$$\rho_2 = -k_\alpha^3 + k_\alpha^2 r_\alpha + \ddot{k}_\alpha - \ddot{r}_\alpha - k_\alpha r_\alpha^2 + r_\alpha^3$$

$$\rho_3 = 2\dot{k}_\alpha r_\alpha - 3r_\alpha \dot{r}_\alpha + k_\alpha \dot{r}_\alpha$$

The torsion of curve γ is

$$r_\gamma = \frac{\sqrt{2}(r_\alpha - k_\alpha)\left[\rho_3(-k_\alpha^2 + k_\alpha r_\alpha) - \rho_1(k_\alpha r_\alpha - r_\alpha^2)\right]}{(k_\alpha - r_\alpha)^2(\dot{k}_\alpha - \dot{r}_\alpha)^2 + (k_\alpha^2 r_\alpha - 2k_\alpha r_\alpha^2 + r_\alpha^3)^2 + (-2k_\alpha^2 r_\alpha + k_\alpha r_\alpha^2 + k_\alpha^3)^2}\,.$$

3.3. Quaternionic $n_1 n_2$-Smarandache Curves.

Definition 6. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternionic curve in E^3 and $\{t_\alpha, n_1^\alpha, n_2^\alpha\}$ be its moving Serret-Frenet frame. Quaternionic $n_1 n_2$-Smarandache curves can be defined by

$$\gamma(s^*) = \frac{1}{\sqrt{2}}(n_1^\alpha + n_2^\alpha) \tag{3.11}$$

Now, we can investigate Serret-Frenet invariants of quaternionic $n_1 n_2$-Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.11) with respect to s, we get

$$\dot{\gamma} = \frac{d\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-k_\alpha t_\alpha - r_\alpha n_1^\alpha + r_\alpha n_2^\alpha) \tag{3.12}$$

and

$$t_\gamma \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(-k_\alpha t_\alpha - r_\alpha n_1^\alpha + r_\alpha n_2^\alpha)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{k_\alpha^2 + 2r_\alpha^2}{2}}\,. \tag{3.13}$$

The tangent vector of curve γ can be written as follow,

$$t_\gamma = \frac{1}{\sqrt{k_\alpha^2 + 2r_\alpha^2}}(-k_\alpha t_\alpha - r_\alpha n_1^\alpha + r_\alpha n_2^\alpha)\,. \tag{3.14}$$

Differentiating (3.14) with respect to s, we obtain

$$\frac{dt_\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{(k_\alpha^2 + 2r_\alpha^2)^{\frac{3}{2}}}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha) \tag{3.15}$$

where

$$\lambda_1 = -k_\alpha^3 r_\alpha - 2\dot{k}_\alpha r_\alpha^2 + 2k_\alpha r_\alpha^3 + 2k_\alpha r_\alpha \dot{r}_\alpha$$

$$\lambda_2 = -k_\alpha^4 - k_\alpha^2 \dot{r}_\alpha - 3k_\alpha^2 r_\alpha^2 - 2r_\alpha^4 + k_\alpha \dot{k}_\alpha r_\alpha$$

$$\lambda_3 = -k_\alpha^2 r_\alpha^2 + k_\alpha^2 \dot{r}_\alpha - 2r_\alpha^4 - k_\alpha \dot{k}_\alpha r_\alpha.$$

Substituting (3.13) in (3.15), we get

$$t_\gamma' = \frac{\sqrt{2}}{(k_\alpha^2 + 2r_\alpha^2)^2}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha)\,.$$

Then, the principal curvature and principal normal vector field of curve γ are respectively,

$$k_\gamma = \|t_\gamma'\| = \frac{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}{(k_\alpha^2 + 2r_\alpha^2)^2}$$

and

$$n_1^\gamma = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha)\,.$$

On the other hand, we express

$$n_2^{\gamma} = -\left\langle V_{t_\gamma}, V_{n_1^\gamma} \right\rangle + V_{t_\gamma} \wedge V_{n_1^\gamma}\,.$$

So, the binormal vector of curve γ is

$$n_2^{\gamma} = \frac{1}{\sqrt{k_\alpha^2 + 2r_\alpha^2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}(\sigma_4 + \sigma_1 t_\alpha + \sigma_2 n_1^\alpha + \sigma_3 n_2^\alpha)$$

where

$$\sigma_1 = -r_\alpha(\lambda_2 + \lambda_3)$$
$$\sigma_2 = k_\alpha \lambda_3 + r_\alpha \lambda_1$$
$$\sigma_3 = -k_\alpha \lambda_2 + r_\alpha \lambda_1$$
$$\sigma_4 = k_\alpha \lambda_1 + r_\alpha \lambda_2 - r_\alpha \lambda_3.$$

We differentiate (3.12) with respect to s in order to calculate the torsion

$$\ddot{\gamma} = \frac{1}{\sqrt{2}}\left[(-\dot{k}_\alpha + k_\alpha r_\alpha)t_\alpha - (k_\alpha^2 + \dot{r}_\alpha + r_\alpha^2)n_1^\alpha + (-r_\alpha^2 + \dot{r}_\alpha)n_2^\alpha\right]$$

and similarly

$$\ddot{\gamma} = \frac{1}{\sqrt{2}}(\rho_1 t_\alpha + \rho_2 n_1^\alpha + \rho_3 n_2^\alpha)$$

where

$$\rho_1 = -\ddot{k}_\alpha + \dot{k}_\alpha r_\alpha + 2k_\alpha \dot{r}_\alpha + k_\alpha^3 + k_\alpha r_\alpha^2$$
$$\rho_2 = -3k_\alpha \dot{k}_\alpha + k_\alpha^2 r_\alpha - \ddot{r}_\alpha - 3r_\alpha \dot{r}_\alpha + r_\alpha^3$$
$$\rho_3 = -k_\alpha^2 r_\alpha - 3r_\alpha \dot{r}_\alpha - r_\alpha^3 + \ddot{r}_\alpha\,.$$

The torsion of curve γ is

$$r_\gamma = \frac{\sqrt{2}\left[\begin{array}{l} r_\alpha(-\dot{k}_\alpha + k_\alpha r_\alpha)(\rho_2 + \rho_3) + (k_\alpha^2 + \dot{r}_\alpha + r_\alpha^2)(k_\alpha \rho_3 + r_\alpha \rho_1) \\ + (-r_\alpha^2 + \dot{r}_\alpha)(k_\alpha \rho_2 - r_\alpha \rho_1) \end{array}\right]}{\left[\begin{array}{l} (k_\alpha \dot{k}_\alpha + 2r_\alpha \dot{r}_\alpha)^2 + (k_\alpha^2 r_\alpha + 2r_\alpha^3)^2 + (k_\alpha \dot{r}_\alpha - \dot{k}_\alpha r_\alpha)^2 \\ + (k_\alpha^3 + k_\alpha \dot{r}_\alpha + 2k_\alpha r_\alpha^2 - \dot{k}_\alpha r_\alpha)^2 \end{array}\right]}\,.$$

3.4. Quaternionic tn_1n_2-Smarandache Curves.

Definition 7. Let $\alpha = \alpha(s)$ be a unit speed spatial quaternionic curve in E^3 and $\{t_\alpha, n_1^\alpha, n_2^\alpha\}$ be its moving Serret-Frenet frame. Quaternionic tn_1n_2-Smarandache curves can be defined by

$$\gamma(s^*) = \frac{1}{\sqrt{3}}(t_\alpha + n_1^\alpha + n_2^\alpha) \qquad (3.16)$$

Now, we can investigate Serret-Frenet invariants of quaternionic

tn_1n_2 -Smarandache curves according to $\alpha = \alpha(s)$. Differentiating (3.16) with respect to s, we get

$$\dot{\gamma} = \frac{d\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-k_\alpha t_\alpha + (k_\alpha - r_\alpha)n_1^\alpha + r_\alpha n_2^\alpha) \tag{3.17}$$

and

$$t_\gamma \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-k_\alpha t_\alpha + (k_\alpha - r_\alpha)n_1^\alpha + r_\alpha n_2^\alpha)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2(k_\alpha^2 - k_\alpha r_\alpha + r_\alpha^2)}{3}}. \tag{3.18}$$

The tangent vector of curve γ can be written as follow,

$$t_\gamma = \frac{1}{\sqrt{2(k_\alpha^2 - k_\alpha r_\alpha + r_\alpha^2)}}(-k_\alpha t_\alpha + (k_\alpha - r_\alpha)n_1^\alpha + r_\alpha n_2^\alpha). \tag{3.19}$$

Differentiating (3.19) with respect to s, we obtain

$$\frac{dt_\gamma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left[2(k_\alpha^2 - k_\alpha r_\alpha + r_\alpha^2)\right]^{\frac{3}{2}}}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha) \tag{3.20}$$

where

$$\lambda_1 = -2k_\alpha^4 + 4k_\alpha^3 r_\alpha + k_\alpha \dot{k}_\alpha r_\alpha - 4k_\alpha^2 r_\alpha^2 - 2\dot{k}_\alpha r_\alpha^2 + 2k_\alpha r_\alpha^3 - k_\alpha^2 \dot{r}_\alpha + 2k_\alpha r_\alpha \dot{r}_\alpha$$

$$\lambda_2 = -2k_\alpha^4 - k_\alpha^2 \dot{r}_\alpha - 4k_\alpha^2 r_\alpha^2 + 2k_\alpha^3 r_\alpha + k_\alpha \dot{k}_\alpha r_\alpha - k_\alpha r_\alpha \dot{r}_\alpha + 2k_\alpha r_\alpha^3 + \dot{k}_\alpha r_\alpha^2 - 2r_\alpha^4$$

$$\lambda_3 = 2k_\alpha^3 r_\alpha - k_\alpha^2 r_\alpha^2 + 2k_\alpha^2 \dot{r}_\alpha + 4k_\alpha r_\alpha^3 - k_\alpha r_\alpha \dot{r}_\alpha - 2r_\alpha^4 - k_\alpha \dot{k}_\alpha r_\alpha + \dot{k}_\alpha r_\alpha^2.$$

Substituting (3.18) in (3.20), we get

$$t_\gamma' = \frac{\sqrt{3}}{\left[2(k_\alpha^2 - k_\alpha r_\alpha + r_\alpha^2)\right]^2}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha).$$

Then, the principal curvature and principal normal vector field of curve γ are respectively,

$$k_\gamma = \left\| t_\gamma' \right\| = \frac{\sqrt{3}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}{\left[2(k_\alpha^2 - k_\alpha r_\alpha + r_\alpha^2)\right]^2}$$

and

$$n_1^\gamma = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}(\lambda_1 t_\alpha + \lambda_2 n_1^\alpha + \lambda_3 n_2^\alpha).$$

On the other hand, we express

$$n_2^\gamma = -\left\langle V_{t_\gamma}, V_{n_1^\gamma} \right\rangle + V_{t_\gamma} \wedge V_{n_1^\gamma}.$$

So, the binormal vector of curve γ is

$$n_2^{\gamma} = \frac{1}{\sqrt{2(k_{\alpha}^2 - k_{\alpha}r_{\alpha} + r_{\alpha}^2)}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}(\sigma_4 + \sigma_1 t_{\alpha} + \sigma_2 n_1^{\alpha} + \sigma_3 n_2^{\alpha})$$

where

$$\begin{aligned} \sigma_1 &= (k_{\alpha} - r_{\alpha})\lambda_3 - r_{\alpha}\lambda_2 \\ \sigma_2 &= k_{\alpha}\lambda_3 + r_{\alpha}\lambda_1 \\ \sigma_3 &= -k_{\alpha}\lambda_2 - (k_{\alpha} - r_{\alpha})\lambda_1 \\ \sigma_4 &= k_{\alpha}\lambda_1 - (k_{\alpha} - r_{\alpha})\lambda_2 - r_{\alpha}\lambda_3. \end{aligned}$$

We differentiate (3.17) with respect to s in order to calculate the torsion

$$\ddot{\gamma} = \frac{1}{\sqrt{3}}\left[(-\dot{k}_{\alpha} - k_{\alpha}^2 + k_{\alpha}r_{\alpha})t_{\alpha} + (-k_{\alpha}^2 + \dot{k}_{\alpha} - \dot{r}_{\alpha} - r_{\alpha}^2)n_1^{\alpha} + (k_{\alpha}r_{\alpha} - r_{\alpha}^2 + \dot{r}_{\alpha})n_2^{\alpha}\right]$$

and similarly

$$\ddot{\gamma} = \frac{1}{\sqrt{3}}(\rho_1 t_{\alpha} + \rho_2 n_1^{\alpha} + \rho_3 n_2^{\alpha})$$

where

$$\begin{aligned} \rho_1 &= -\ddot{k}_{\alpha} - 3k_{\alpha}\dot{k}_{\alpha} + \dot{k}_{\alpha}r_{\alpha} + 2k_{\alpha}\dot{r}_{\alpha} + k_{\alpha}^3 + k_{\alpha}r_{\alpha}^2 \\ \rho_2 &= -3k_{\alpha}\dot{k}_{\alpha} - k_{\alpha}^3 + k_{\alpha}^2 r_{\alpha} + \ddot{k}_{\alpha} - \ddot{r}_{\alpha} - 3r_{\alpha}\dot{r}_{\alpha} - k_{\alpha}r_{\alpha}^2 + r_{\alpha}^3 \\ \rho_3 &= -k_{\alpha}^2 r_{\alpha} + 2\dot{k}_{\alpha}r_{\alpha} - 3r_{\alpha}\dot{r}_{\alpha} - r_{\alpha}^3 + k_{\alpha}\dot{r}_{\alpha} + \ddot{r}_{\alpha}. \end{aligned}$$

The torsion of curve γ is

$$r_{\gamma} = \frac{\sqrt{3}\begin{bmatrix} (-\dot{k}_{\alpha} - k_{\alpha}^2 + k_{\alpha}r_{\alpha})\left((r_{\alpha} - k_{\alpha})\rho_3 + r_{\alpha}\rho_2\right) \\ + (k_{\alpha}^2 - \dot{k}_{\alpha} + \dot{r}_{\alpha} + r_{\alpha}^2)(k_{\alpha}\rho_3 + r_{\alpha}\rho_1) \\ + (k_{\alpha}r_{\alpha} - r_{\alpha}^2 + \dot{r}_{\alpha})(k_{\alpha}\rho_2 + (k_{\alpha} - r_{\alpha})\rho_1) \end{bmatrix}}{\begin{bmatrix} (-2k_{\alpha}\dot{k}_{\alpha} + \dot{k}_{\alpha}r_{\alpha} - 2r_{\alpha}\dot{r}_{\alpha} + k_{\alpha}\dot{r}_{\alpha})^2 + (2k_{\alpha}^2 r_{\alpha} - 2k_{\alpha}r_{\alpha}^2 + k_{\alpha}\dot{r}_{\alpha} + 2r_{\alpha}^3 - \dot{k}_{\alpha}r_{\alpha})^2 \\ + (k_{\alpha}\dot{r}_{\alpha} - \dot{k}_{\alpha}r_{\alpha})^2 + (k_{\alpha}^3 + k_{\alpha}\dot{r}_{\alpha} + 2k_{\alpha}r_{\alpha}^2 - \dot{k}_{\alpha}r_{\alpha} - 2k_{\alpha}^2 r_{\alpha} + k_{\alpha}^3)^2 \end{bmatrix}}.$$

References

[1] A. C. Çöken and A. Tuna, On the Quaternionic Inclined Curves in the Semi-Euclidean Space E_2^4, Applied Mathematics and Computation, 155 (2004), 373-389.

[2] A. T. Ali, Special Smarandache Curves in the Euclidean Space, International Journal of Mathematical Combinatorics, 2 (2010), 30-36.

[3] F. Kahraman, İ. Gök and H. H. Hacısalihoğlu, On the Quaternionic B_2-Slant Helices in the Semi-Euclidean Space E_2^4, Applied Mathematics and Computation, 218 (2012), 6391-6400.

[4] H. H. Hacısalihoğlu, Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi, Fen-Edebiyat Fakültesi Yayınları Mat., No.2, 1983.

[5] İ. Gök, O. Z. Okuyucu, F. Kahraman and H. H. Hacısalihoğlu, On the Quaternionic B_2-Slant Helices in the Euclidean Space E^4, Advances in Applied Clifford Algebras, 21 (2011), 707-719.

[6] K. Bharathi and M. Nagaraj, Quaternion Valued Function of a Real Serret-Frenet Formulae, Indian J. Pure Appl. Math., 16 (1985), 741-756.

[7] M. A. Güngör and M. Tosun, Some Characterizations of Quaternionic Rectifying Curves, Differential Geometry - Dynamical Systems, 13 (2011), 89-100.

[8] M. Çetin, Y. Tunçer and M. K. Karacan, Smarandache Curves According to Bishop Frame in Euclidean 3-Space, arXiv:1106.3202.v1 [math. DG], 16. Jun 2011.

[9] M. Önder, Quaternionic Salkowski Curves and Similar Curves, arXiv:1205.1368v1 [math.DG], 7 May 2012.

[10] M. Önder, Quaternionic Smilar Curves in 4-dimensional Euclidean space E^4, arXiv:1205.4137v1 [math. DG], 18 May 2012.

[11] J. P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publishers, Boston/London, 1997.

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