On the k-Riemann-Liouville Fractional Derivative

Luis Guillermo Romero¹, Luciano L. Luque,
Gustavo Abel Dorrego and Rubén A. Cerutti²

Faculty of Exact Sciences
National University of Nordeste. Av. Libertad 5540
(3400) Corrientes, Argentina
¹e-mail: guille-romero@live.com.ar
²e-mail: rcerutti@exa.unne.edu.ar

Abstract

The main object of this paper is to introduce a new fractional operator called k-Riemann-Liouville fractional derivative defined by using the k-Gamma function, which is a generalization of the classical Gamma function. We also investigate relationships with the k-Riemann-Liouville integral and derive some properties by using Fourier and Laplace transform.

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I Introduction and Preliminaries

Since the k-Gamma function and the k-Pochhammer symbol introduced by Diaz y Pariguan (cf. [2]) the fractional calculus has been significant development through the generalization of the so-called special functions, like the k-Beta, the k-Mittag-Leffler function, the k-Wright function, the k-Bessel functions. (cf. [1], [3], [7], [9]).

Also a k-fractional integral of the Riemann-Liouville type has been introduced in 2012. (cf [5]).

For the development of our work, recalling the definition of the Riemann-Liouville fractional integral we have

Definition 1 Let $f$ be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let $\nu$ be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order $\nu$, $I_+^{\nu}f$ is given by
\[ I_\nu^+ f(t) = \frac{1}{\Gamma(\nu)} \int_{a}^{t} (t - \tau)^{\nu-1} f(\tau) d\tau \]  

(I.1)

where \( \Gamma(z) \) is the Euler Gamma function

\[ \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt \]

for \( \text{Re}(z) > 0, \ z \) a complex number.

As a particular case of (I.1), taking \( a = 0 \) results

\[ I_\nu^+ f(t) = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t - \tau)^{\nu-1} f(\tau) d\tau \]  

(I.2)

The Riemann-Liouville fractional derivative of order \( \nu > 0 \), \( D_+^\nu \) is defined as the left inverse of the Riemann-Liouville integral of order \( \nu \); i.e.,

\[ D_+^\nu I_\nu^+ f(t) = I_d f(t), \ \nu > 0 \quad \text{cf. [10]} \]  

(I.3)

Another way to defined this fractional derivative is as follows.

**Definition 2** Let \( \nu \) be a real number, and let \( m \) be the integer sucht that \( m - 1 < \nu \leq m \). Then the Riemann-Liouville fractional derivative of order \( \nu \) is given by

\[ D_+^\nu f(t) = D^m I_\nu^+ f(t) \]  

(I.4)

Equivalently, we have

\[ D_+^\nu f(t) = \left\{ \begin{array}{ll}
\frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\nu)} \int_{0}^{t} \frac{f(\tau)d\tau}{(t-\tau)^{m-\nu+1}} \right], & m - 1 < \nu \leq m \\
\frac{d^m}{dt^m} f(t), & \nu = m
\end{array} \right. \]  

(I.5)

Let \( \mathcal{L}(f)(z) \) be the Laplace transform of an exponential order function and piecewise continuous given by

\[ \mathcal{L}(f)(z) = \int_{0}^{\infty} e^{-zt} f(t) dt \]  

(I.6)

\( t \in \mathbb{R}^+ \), and \( z \in \mathbb{C} \).
And let $\mathcal{F}[\varphi]$ be the Fourier Transform of the function $\varphi$ belonging to $S(\mathbb{R})$ the Schwartzian space of functions that decay rapidly at infinity together with all derivatives, defined by the integral

$$
\mathcal{F}[\varphi](x) = \int_{-\infty}^{+\infty} e^{ixt} \varphi(t) \, dt \quad (I.7)
$$

Relationships between these integral transforms and the Riemann-Liouville operators are listed below.

For the Riemann-Liouville fractional integral we have

$$
\mathcal{L} \left[ I^{\nu}_+ f \right] (z) = \frac{\mathcal{L} (f) (z)}{z^{\nu}} \quad \text{cf. [6]} \quad (I.8)
$$

and for the Riemann-Liouville fractional derivative

$$
\mathcal{L} \left[ D^{\nu}_+ f \right] (z) = z^{\nu} \mathcal{L} (f) (z) - \sum_{k=0}^{m-1} \left[ D^k I^{m-k}_+ \right] f(0^+)z^{m-k-1}; \quad m - 1 < k \leq m \quad (I.9)
$$

cf.[6].

Also, the Fourier transform of the Riemann-Liouville fractional integral is given by

$$
\mathfrak{F} \left[ I^{\nu}_+ f \right] (x) = \frac{\mathfrak{F} (f) (x)}{(-ix)^{\nu}} \quad (I.10)
$$

and for the Riemann-Liouville fractional derivative

$$
\mathfrak{F} \left[ D^{\nu}_+ f \right] (x) = (-ix)^{\nu} \mathfrak{F} (f) (x) \quad (I.11)
$$

cf.[4], cf.[6].

Now we introduced the k-Riemann-Liouville fractional integral by the following

**Definition 3** Let $f$ be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let $\nu$ be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order $\nu$, $I^{\nu}_+ f$ is given by

$$
I^{\alpha}_{k,a} f(x) = \frac{1}{k \Gamma_k (\alpha)} \int_{a}^{x} (x - t)^{\frac{\alpha}{k} - 1} f(t) \, dt \quad (I.12)
$$
As special case can be obtained the definition due to Mubeem and G. Habibullah (cf. [5])

\[
I_k^\alpha f(t) = \frac{1}{k \Gamma_k(\alpha)} \int_0^x (x - t)^{\frac{\alpha}{k} - 1} f(t) dt
\]  

(I.13)

where \( \Gamma_k(z) \) denote the k-Gamma function given (cf. [2]) by

\[
\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt
\]

Romero, Cerutti and Dorrego introduced the k-Riemann-Liouville singular kernel given (cf. [8]) the following

**Definition 4** Let \( \alpha \) be a real number, \( 0 < \alpha < 1 \), \( k > 0 \). The k-Riemann-Liouville singular kernel is given by

\[
j_{\alpha,k}(t) = \frac{t^{\frac{\alpha}{k}-1}}{k \Gamma_k(\alpha)} \quad t > 0
\]  

(I.14)

It can be proved that the k-Riemann-Liouville fractional integral may be expressed as the convolution

\[
I_k^\alpha f(x) = j_{\alpha,k}(t) * f(t)
\]  

(I.15)

**II Main Result**

This section presents results linking k-fractional integral with Fourier integral transform and Laplace transform as well as some examples of application.

For the development of our work we need to remember the following

**Definition 5** Let \( \beta \) be a real number, \( 0 < \beta \leq 1 \). The k-Riemann-Liouville fractional derivative is given by

\[
D_k^\beta f(t) = \frac{d}{dt} I_k^{1-\beta} f(t)
\]  

(II.1)

Easily it can be proved that \( I_k^{1-\beta} (j_{\alpha,k}(t)) = j_{\alpha+1-\beta,k}(t) \) and \( \frac{d}{dt} j_{\alpha,k}(t) = \frac{1}{k} j_{\alpha-k,k}(t) \) then we have

\[
D_k^\beta (j_{\alpha,k}(t)) = \frac{1}{k} j_{\alpha+1-\beta-k,k}(t)
\]  

(II.2)

However, this operator is not a left inverse operator with respect the k-Riemann-Liouville fractional integral operator. In fact,
Let $0 < \alpha < 1$. From definition (II.1) we have

$$D^{\beta}_{k}I^{\beta}_{k}f(t) = \frac{d}{dt} \left[ I^{1-\beta}_{k} \left( I^{\beta}_{k}f(t) \right) \right] = \frac{d}{dt} I^{1}_{k}f(t) =$$

$$= \frac{1}{k\Gamma_{k}(1)} \frac{d}{dt} \int_{0}^{t} (t-\tau)^{\frac{1}{k}-1}f(\tau)d\tau = \frac{1}{k\Gamma_{k}(1)} \frac{1-k}{k} \int_{0}^{t} (t-\tau)^{\frac{1}{k}-1}f(\tau)d\tau =$$

$$= \frac{1-k}{k\Gamma_{k}(1)} \frac{\Gamma_{k} \left( \frac{1-k}{k} \right)}{k\Gamma_{k} \left( \frac{1-k}{k} \right)} \int_{0}^{t} (t-\tau)^{\frac{1-k}{k}-1}f(\tau)d\tau =$$

$$= (1-k) \Gamma_{k} \left( \frac{1-k}{k} \right) \frac{1}{k\Gamma_{k}(1)} I^{1-\beta}_{k}f(t) \neq f(t)$$

In the two next lemmas we will evaluate the Laplace and Fourier transform of the k-Riemann-Liouville fractional integral.

**Lemma 1** Let $f$ be a sufficiently well-behaved function and let $\alpha$ be a real number, $0 < \alpha \leq 1$. The Laplace transform of the k-Riemann-Liouville fractional integral of the $f$ function is given by

$$\mathcal{L} \left[ I^{\alpha}_{k}f \right] (s) = (ks)^{-\alpha} \mathcal{L} \left[ f \right] (s)$$

(II.3)

**Proof.**

First, we evaluate the Laplace Transform of the k-Riemann-Liouville singular kernel.

$$\mathcal{L} \left[ j_{\alpha,k}(t) \right] (s) = \int_{0}^{+\infty} t^{\frac{1}{k}-1}e^{-st}dt =$$

$$= \frac{1}{k\Gamma_{k}(\alpha)} \int_{0}^{+\infty} t^{\frac{1}{k}-1}e^{-st}dt$$

(II.4)

Taking into account that the integral in (II.4) is

$$\int_{0}^{+\infty} t^{\frac{1}{k}-1}e^{-st}dt = \frac{\Gamma \left( \frac{\alpha}{k} \right)}{s^{\alpha/k}}$$

(II.5)

From (II.5) and (II.4) we have

$$\mathcal{L} \left[ j_{\alpha,k}(t) \right] (s) = \frac{1}{k\Gamma_{k}(\alpha)} \frac{\Gamma \left( \frac{\alpha}{k} \right)}{s^{\alpha/k}} =$$
\[
\frac{1}{k,k^{\frac{\alpha}{k}}-\Gamma}\frac{\Gamma\left(\frac{\alpha}{k}\right)}{s^{\frac{\alpha}{k}}} = (ks)^{-\frac{\alpha}{k}} \quad \text{(II.6)}
\]

We known that the Laplace transform applied the convolution product to a point wise product, then it results
\[
\mathcal{L} [I_k^n f] (s) = \mathcal{L} [j_{\alpha,k} * f] (s) = \mathcal{L} [j_{\alpha,k}] (s) \cdot \mathcal{L} [f] (s) \quad \text{(II.7)}
\]

Finally, from (II.7) and (II.6) we obtain
\[
\mathcal{L} [I_k^n f] (s) = (ks)^{-\frac{\alpha}{k}} \mathcal{L} [f] (s) \quad \text{(II.8)}
\]

**Lemma 2** Let \( f \) be a sufficiently well-behaved function and let \( \alpha \) be a real number, \( 0 < \alpha \leq 1 \). The Fourier transform of the \( k \)-Riemann-Liouville fractional integral of the \( f \) function is
\[
\mathcal{F} [I_k^n f] (\omega) = (-ik\omega)^{-\frac{\alpha}{k}} \mathcal{F} [f] (\omega) \quad \text{(II.9)}
\]

**Proof.**

The Fourier transform verified a property analogous to the Laplace transform with respect the convolution product. In fact
\[
\mathcal{F} [I_k^n f] (\omega) = \mathcal{F} [j_{\alpha,k} * f] (\omega) = \mathcal{F} [j_{\alpha,k}] (\omega) \cdot \mathcal{F} [f] (\omega) \quad \text{(II.10)}
\]

By evaluating the Fourier transform of the \( k \)-Riemann-Liouville singular kernel
\[
\mathcal{F} [j_{\alpha,k}(t)] (\omega) = \int_0^{+\infty} \frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} e^{i\omega t} dt \quad \text{(II.11)}
\]
and making the change of variables \(-s = i\omega\) it result
\[
\mathcal{F} [j_{\alpha,k}(t)] (\omega) = \int_0^{+\infty} \frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} e^{-st} dt =
\]
\[
\mathcal{L} [j_{\alpha,k}(t)] (s) = (ks)^{-\frac{\alpha}{k}} = (-i\omega k)^{-\frac{\alpha}{k}} \quad \text{(II.12)}
\]

Replacing (II.12) in (II.10) we have
\[
\mathcal{F} [I_k^n f] (\omega) = (-i\omega k)^{-\frac{\alpha}{k}} \mathcal{F} [f] (\omega) \quad \text{(II.13)}
\]
We also will use the following properties of the Fourier and Laplace Transform

\[ \mathcal{L} \left[ \frac{d}{dt} f(t) \right] (s) = s \mathcal{L} [f(t)] (s) - f(0) \] (II.14)

and

\[ \mathcal{F} \left[ \frac{d}{dt} f(t) \right] (\omega) = (-i\omega) \mathcal{F} [f(t)] (\omega) \] (II.15)

To obtain the Fourier and Laplace transform of the \( k \)-Riemann-Liouville fractional derivative.

**Lemma 3** Let \( \beta \) be a real number, \( 0 < \beta \leq 1 \). The Laplace transform of the \( k \)-Riemann-Liouville fractional derivative is given by

\[ \mathcal{L} \left[ D^\beta_k f \right] (s) = s. (ks)^{-\frac{1-\beta}{k}} L[f](s) - I^1_1 f(0) \] (II.16)

**Proof.**

By definition (II.1) and property (II.14) we arrive at

\[ \mathcal{L} \left[ D^\beta_k f \right] (s) = \mathcal{L} \left[ \frac{d}{dt} I^1_1 f(t) \right] (s) = \]

\[ s \mathcal{L} \left[ I^1_1 f(t) \right] (s) - I^1_1 f(0) \] (II.17)

From (II.8) and (II.17) we have

\[ \mathcal{L} \left[ D^\beta_k f \right] (s) = s. (ks)^{-\frac{1-\beta}{k}} L[f](s) - I^1_1 f(0) \]

**Lemma 4** Let \( \beta \) be a real number, \( 0 < \beta \leq 1 \). The Fourier transform of the \( k \)-Riemann-Liouville fractional derivative is given by

\[ \mathcal{F} \left[ D^\beta_k f \right] (\omega) = (-i\omega). (-i\omega k)^{-\frac{1-\beta}{k}} \mathcal{F} [f] (\omega) \] (II.18)

**Proof.**

By definition (II.1) and property (II.15) we have

\[ \mathcal{F} \left[ D^\beta_k f(t) \right] (\omega) = \mathcal{F} \left[ \frac{d}{dt} I^1_1 f(t) \right] (\omega) = \]
\[ (-i\omega) \mathcal{F} \left[ I_k^{1-\beta} f(t) \right] (\omega) \]  

(II.19)

From (II.13) and (II.19) we have

\[ \mathcal{F} \left[ D_k^\beta f \right] (\omega) = (-i\omega) \cdot (-i\omega k)^{-\frac{1-\beta}{k}} \mathcal{F} [f] (\omega) \]

Examples of application

Example 1

\( (D_{k,a}^{\alpha} (t-a)^{\beta-1}) (x) = \frac{k^{1-n} \Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta-kn)} (x-a)^{\frac{n-\alpha}{k}+\beta-(n+1)} \)

Proof.

\[ (D_{k,a}^{\alpha} (t-a)^{\beta-1}) (x) = \left( \frac{d}{dx} \right)^n (I_{k,a}^{n-\alpha} (t-a)^{\beta-1}) (x) \]  

(II.20)

In (II.20), calling \( p = n - \alpha \) we calculate

\[ (I_{k,a}^{n-\alpha} (t-a)^{\beta-1}) (x) = (I_{k,a}^p (t-a)^{\beta-1}) (x) = \]

\[ \frac{1}{k \Gamma_k(p)} \int_a^x (x-t)^{\frac{p}{k}-1}(t-a)^{\beta-1}dt \]  

(II.21)

Making the change of variables \( t = a + \varepsilon (x-a) \) it result

\[ (I_{k,a}^{n-\alpha} (t-a)^{\beta-1}) (x) = \frac{1}{k \Gamma_k(p)} \int_0^1 [(1-\varepsilon)((x-a))]^{\frac{p}{k}-1} [\varepsilon(x-a)]^{\beta-1} (x-a)d\varepsilon = \]

\[ \frac{1}{\Gamma_k(p)} (x-a)^{\frac{p}{k}+\beta-1} \frac{1}{k} \int_0^1 (1-\varepsilon)^{\frac{p}{k}-1} \varepsilon^{\beta-1}d\varepsilon = \]

\[ \frac{(x-a)^{\frac{p}{k}+\beta-1}}{\Gamma_k(p)} B_k(p, k\beta) \]  

(II.22)

where \( B_k(x, y) \) denote the k-Beta function.

Remembering that \( B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \), it result
\[(I_{k,a}^{n-\alpha}(t-a)^{\beta-1})(x) = \frac{(x-a)^{\frac{p}{k}+\beta-1} \Gamma_k(p) \Gamma_k(k\beta)}{\Gamma_k(p+k\beta)} = \frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)}(x-a)^{\frac{n-\alpha}{k}+\beta-1} \quad (\text{II.23})\]

Replacing (II.23) in (II.20) we have
\[(D_{k,a}^\alpha(t-a)^{\beta-1})(x) = \frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)} \left( \frac{d}{dx} \right)^n (x-a)^{\frac{n-\alpha}{k}+\beta-1} = \]
\[
\frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)} \frac{\Gamma_k(n-\alpha+k\beta)}{\Gamma_k(n-\alpha+k\beta-kn)} k^{\frac{1-n}{k}}(x-a)^{\frac{n-\alpha}{k}+\beta-(n+1)} \quad (\text{II.24})
\]

**Example 2**

\[(D_{k,a}^\alpha 1)(x) = \frac{k^{1-n}(x-a)^{\frac{n-\alpha}{k}-n}}{\Gamma_k(n-\alpha+k(1-n))} \]

**Proof.**
It is sufficient take \(\beta = 1\) in Example 1.

**Example 3**

\[(D_{k,a}^\alpha(t-a)^{\alpha-j})(x) = \frac{k^{1-n}\Gamma_k(n\alpha-kj+k)}{\Gamma_k(n-\alpha+k\alpha-jk+k-kn)} (x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}} \]

**Proof.**
\[(D_{k,a}^\alpha(t-a)^{\alpha-j})(x) = \left( \frac{d}{dx} \right)^n (I_{k,a}^{n-\alpha}(t-a)^{\alpha-j})(x) \quad (\text{II.25})\]

In (II.25), calling \(p = n-\alpha\) we calculate
\[(I_{k,a}^p(t-a)^{\alpha-j})(x) = \frac{1}{k\Gamma_k(p)} \int_a^x (x-t)^{\frac{p}{k}-1}(t-a)^{\alpha-j}dt \quad (\text{II.26})\]
Making the change of variables \( t = a + \varepsilon(x - a) \) it result

\[
(I^p_{k,a}(t-a)^{\alpha-j})(x) = \frac{1}{k\Gamma_k(p)} \int_0^1 [(1-\varepsilon)((x-a))]^{\frac{p}{k}-1} [\varepsilon(x-a)]^{\alpha-j} (x-a)d\varepsilon =
\]

\[
\frac{1}{\Gamma_k(p)}(x-a)^{\frac{p}{k}+\alpha-j} \frac{1}{k} \int_0^1 (1-\varepsilon)^{\frac{p}{k}-1} \varepsilon^{k(\alpha-j+1)} d\varepsilon =
\]

\[
\frac{(x-a)^{\frac{p}{k}+\alpha-j}}{\Gamma_k(p)} B_k(p, k(\alpha-j+1))
\]

\[
\frac{(x-a)^{\frac{p}{k}+\alpha-j} \Gamma_k(p) \Gamma_k(k(\alpha-j+1))}{\Gamma_k(p) \Gamma_k(p+k(\alpha-j+1))} =
\]

\[
\frac{\Gamma_k(k(\alpha-j+1))}{\Gamma_k(n-\alpha+k(\alpha-j+1))} (x-a)^{\frac{n-\alpha}{k}+\alpha-j}
\]

(II.27)

Replacing (II.27) in (II.25) we have

\[
(D^\alpha_{k,a}(t-a)^{\alpha-j})(x) = \frac{\Gamma_k(k(\alpha-j+1))}{\Gamma_k(n-\alpha+k(\alpha-j+1))} \left( \frac{d}{dx} \right)^n (x-a)^{\frac{n-\alpha}{k}+\alpha-j} =
\]

\[
\frac{\Gamma_k(n\alpha - kj + k)}{\Gamma_k(n-\alpha+k\alpha-jk+k-kn)} k^{1-n} (x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}}
\]

(II.28)

Example 4

\[
D^\alpha_{k,a} \left[ \sum_{j=1}^n c_j(x-a)^{\alpha-j} \right] (x) =
\]

\[
\sum_{j=1}^n c_j \left( \frac{\Gamma_k(n\alpha - kj + k)}{\Gamma_k(n-\alpha+k\alpha-jk+k-kn)} k^{1-n} (x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}} \right)
\]

Proof.

From (II.28) we have the example 4.


References


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