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On the k-Riemann-Liouville Fractional Derivative

Luis Guillermo Romero[1], Luciano L. Luque, Gustavo Abel Dorrego and Rubén A. Cerutti[2]

Faculty of Exact Sciences
National University of Nordeste. Av. Libertad 5540
(3400) Corrientes, Argentina
[1]e-mail: guille-romero@live.com.ar
[2]e-mail: rcerutti@exa.unne.edu.ar

Abstract

The main object of this paper is to introduce a new fractional operator called k-Riemann-Liouville fractional derivative defined by using the k-Gamma function, which is a generalization of the classical Gamma function. We also investigate relationships with the k-Riemann-Liouville integral and derive some properties by using Fourier and Laplace transform.

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I Introduction and Preliminaries

Since the k-Gamma function and the k-Pochhammer symbol introduced by Diaz y Pariguan (cf. [2])the fractional calculus has been significant development through the generalization of the so-called special functions, like the k-Beta, the k-Mittag-Leffler function, the k-Wright function, the k-Bessel functions. (cf. [1], [3], [7], [9]).

Also a k-fractional integral of the Riemann-Liouville type has been introduced in 2012.(cf [5]).

For the development of our work, recalling the definition of the Riemann-Liouville fractional integral we have

Definition 1 *Let f be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let ν be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order ν, $I_+^\nu f$ is given by*

$$I_+^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\tau)^{\nu-1} f(\tau) d\tau \tag{I.1}$$

where $\Gamma(z)$ *is the Euler Gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for $Re(z) > 0$*,* z *a complex number.*

As a particular case of (I.1), taking $a = 0$ results

$$I_+^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau \tag{I.2}$$

The Riemann-Liouville fractional derivative of order $\nu > 0$, D_+^ν is defined as the left inverse of the Riemann-Liouville integral of order ν; i. e,

$$D_+^\nu I_+^\nu = Id, \quad \nu > 0 \quad \text{cf. [10]} \tag{I.3}$$

Another way to defined this fractional derivative is as follows.

Definition 2 *Let* ν *be a real number, and let* m *be the integer sucht that* $m-1 < \nu \leq m$*. Then the Riemann-Liouville fractional derivative of order* ν *is given by*

$$D_+^\nu f(t) = D^m I_+^{m-\nu} f(t) \tag{I.4}$$

Equivalently, we have

$$D_+^\nu f(t) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\nu)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\nu+1-m}}\right], & m-1 < \nu \leq m \\ \frac{d^m}{dt^m} f(t) & , \quad \nu = m \end{cases} \tag{I.5}$$

Let $\mathfrak{L}(f)(z)$ be the Laplace transform of an exponential order function and piecewise continuous given by

$$\mathfrak{L}(f)(z) = \int_0^\infty e^{-zt} f(t) dt \tag{I.6}$$

$t \epsilon \mathbb{R}^+$, and $z \epsilon \mathbb{C}$.

And let $\mathfrak{F}[\varphi]$ be the Fourier Transform of the function φ belonging to $S(\mathbb{R})$ the Schwartzian space of functions that decay rapidly at infinity together with all derivatives, defined by the integral

$$\mathfrak{F}[\varphi](x) = \int_{-\infty}^{+\infty} e^{ixt}\varphi(t)dt \tag{I.7}$$

Relationships between these integral transforms and the Riemann-Liouville operators are listed below.

For the Riemann-Liouville fractional integral we have

$$\mathfrak{L}\left[I_+^\nu f\right](z) = \frac{\mathfrak{L}(f)(z)}{z^\nu} \quad \text{cf. [6]} \tag{I.8}$$

and for the Riemann-Liouville fractional derivative

$$\mathfrak{L}\left[D_+^\nu f\right](z) = z^\nu \mathfrak{L}(f)(z) - \sum_{k=0}^{m-1}\left[D^k I_+^{m-k}\right] f(0^+) z^{m-k-1}; \quad m-1 < k \le m \tag{I.9}$$

cf.[6].

Also, the Fourier transform of the Riemann-Liouville fractional integral is given by

$$\mathfrak{F}\left[I_+^\nu f\right](x) = \frac{\mathfrak{F}(f)(x)}{(-ix)^\nu} \tag{I.10}$$

and for the Riemann-Liouville fractional derivative

$$\mathfrak{F}\left[D_+^\nu f\right](x) = (-ix)^\nu \mathfrak{F}(f)(x) \tag{I.11}$$

cf.[4], cf.[6].

Now we introduced the k-Riemann-Liouville fractional integral by the following

Definition 3 *Let f be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let ν be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order ν, $I_+^\nu f$ is given by*

$$I_{k,a}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt \tag{I.12}$$

As special case can be obtained the definition due to Mubeem and G. Habibullah (cf. [5])

$$I_k^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt \tag{I.13}$$

where $\Gamma_k(z)$ denote the k-Gamma function given (cf. [2]) by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt$$

Romero, Cerutti and Dorrego introduced the k-Riemann-Liouville singular kernel given (cf. [8]) the following

Definition 4 *Let α be a real number, $0 < \alpha < 1$, $k > 0$. The k-Riemann-Liouville singular kernel is given by*

$$j_{\alpha,k}(t) = \frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} \quad t > 0 \tag{I.14}$$

It can be proved that the k-Riemann-Liouville fractional integral may be expressed as the convolution

$$I_k^{\alpha} f(x) = j_{\alpha,k}(t) * f(t) \tag{I.15}$$

II Main Result

This section presents results linking k-fractional integral with Fourier integral transform and Laplace transform as well as some examples of application.

For the development of our work we need to remember the following

Definition 5 *Let β be a real number, $0 < \beta \leq 1$. The k-Riemann-Liouville fractional derivative is given by*

$$D_k^{\beta} f(t) = \frac{d}{dt} I_k^{1-\beta} f(t) \tag{II.1}$$

Easily it can be proved that $I_k^{1-\beta}(j_{\alpha,k}(t)) = j_{\alpha+1-\beta,k}(t)$ and $\frac{d}{dt} j_{\alpha,k}(t) = \frac{1}{k} j_{\alpha-k,k}(t)$ then we have

$$D_k^{\beta}(j_{\alpha,k}(t)) = \frac{1}{k} j_{\alpha+1-\beta-k,k}(t) \tag{II.2}$$

However, this operator is not a left inverse operator with respect the k-Riemann-Liouville fractional integral operator. In fact, Let $0 < \alpha < 1$. From definition (II.1) we have

$$D_k^\beta I_k^\beta f(t) = \frac{d}{dt}\left[I_k^{1-\beta}\left(I_k^\beta f(t)\right)\right] = \frac{d}{dt} I_k^1 f(t) =$$

$$= \frac{1}{k\Gamma_k(1)}\frac{d}{dt}\int_0^t (t-\tau)^{\frac{1}{k}-1} f(\tau)d\tau = \frac{1}{k\Gamma_k(1)}\frac{1-k}{k}\int_0^t (t-\tau)^{\frac{1-k}{k}-1} f(\tau)d\tau =$$

$$= \frac{1-k}{k\Gamma_k(1)}\frac{\Gamma_k\left(\frac{1-k}{k}\right)}{k\Gamma_k\left(\frac{1-k}{k}\right)}\int_0^t (t-\tau)^{\frac{1-k}{k}-1} f(\tau)d\tau$$

$$= \frac{(1-k)\Gamma_k\left(\frac{1-k}{k}\right)}{k\Gamma_k(1)} I_k^{\frac{1-k}{k}} f(t) \neq f(t)$$

In the two next lemmas we will evaluate the Laplace and Fourier transform of the k-Riemann-Liouville fractional integral.

Lemma 1 *Let f be a sufficiently well-behaved function and let α be a real number, $0 < \alpha \leq 1$. The Laplace transform of the k-Riemann-Liouville fractional integral of the f function is given by*

$$\mathfrak{L}\left[I_k^\alpha f\right](s) = (ks)^{-\frac{\alpha}{k}}\,\mathfrak{L}\left[f\right](s) \tag{II.3}$$

Proof.

First, we evaluate the Laplace Transform of the k-Riemann-Liouville singular kernel.

$$\mathfrak{L}\left[j_{\alpha,k}(t)\right](s) = \int_0^{+\infty} \frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} e^{-st}dt =$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_0^{+\infty} t^{\frac{\alpha}{k}-1} e^{-st}dt \tag{II.4}$$

Taking into account that the integral in (II.4) is

$$\int_0^{+\infty} t^{\frac{\alpha}{k}-1} e^{-st}dt = \frac{\Gamma\left(\frac{\alpha}{k}\right)}{s^{\frac{\alpha}{k}}} \tag{II.5}$$

From (II.5) and (II.4) we have

$$\mathfrak{L}\left[j_{\alpha,k}(t)\right](s) = \frac{1}{k\Gamma_k(\alpha)}\frac{\Gamma\left(\frac{\alpha}{k}\right)}{s^{\frac{\alpha}{k}}} =$$

$$\frac{1}{k.k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)}\frac{\Gamma\left(\frac{\alpha}{k}\right)}{s^{\frac{\alpha}{k}}}=(ks)^{-\frac{\alpha}{k}} \tag{II.6}$$

We known that the Laplace transform applied the convolution product to a point wise product, then it results

$$\mathfrak{L}\left[I_k^\alpha f\right](s)=\mathfrak{L}\left[j_{\alpha,k}*f\right](s)=\mathfrak{L}\left[j_{\alpha,k}\right](s).\mathfrak{L}[f](s) \tag{II.7}$$

Finally, from (II.7) and (II.6) we obtain

$$\mathfrak{L}\left[I_k^\alpha f\right](s)=(ks)^{-\frac{\alpha}{k}}\,\mathfrak{L}[f](s) \tag{II.8}$$

Lemma 2 *Let f be a sufficiently well-behaved function and let α be a real number, $0<\alpha\leq 1$. The Fourier transform of the k-Riemann-Liouville fractional integral of the f function is*

$$\mathfrak{F}\left[I_k^\alpha f\right](\omega)=(-ik\omega)^{-\frac{\alpha}{k}}\,\mathfrak{F}\left[f\right](\omega) \tag{II.9}$$

Proof.

The Fourier transform verified a property analogous to the Laplace transform with respect the convolution product. In fact

$$\mathfrak{F}\left[I_k^\alpha f\right](\omega)=\mathfrak{F}\left[j_{\alpha,k}*f\right](\omega)=\mathfrak{F}\left[j_{\alpha,k}\right](\omega).\mathfrak{F}[f](\omega) \tag{II.10}$$

By evaluating the Fourier transform of the k-Riemann-Liouville singular kernel

$$\mathfrak{F}\left[j_{\alpha,k}(t)\right](\omega)=\int_0^{+\infty}\frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}e^{i\omega t}dt \tag{II.11}$$

and making the change of variables $-s=i\omega$ it result

$$\mathfrak{F}\left[j_{\alpha,k}(t)\right](\omega)=\int_0^{+\infty}\frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}e^{-st}dt=$$

$$\mathfrak{L}\left[j_{\alpha,k}(t)\right](s)=(ks)^{-\frac{\alpha}{k}}=(-i\omega k)^{-\frac{\alpha}{k}} \tag{II.12}$$

Replacing (II.12) in (II.10) we have

$$\mathfrak{F}\left[I_k^\alpha f\right](\omega)=(-i\omega k)^{-\frac{\alpha}{k}}\,\mathfrak{F}[f](\omega) \tag{II.13}$$

We also will use the following properties of the Fourier and Laplace Transform

$$\mathfrak{L}\left[\frac{d}{dt}f(t)\right](s) = s\mathfrak{L}\left[f(t)\right](s) - f(0) \tag{II.14}$$

and

$$\mathfrak{F}\left[\frac{d}{dt}f(t)\right](\omega) = (-i\omega)\,\mathfrak{F}\left[f(t)\right](\omega) \tag{II.15}$$

To obtain the Fourier and Laplace transform of the k-Riemann-Liouville fractional derivative.

Lemma 3 *Let β be a real number, $0 < \beta \leq 1$. The Laplace transform of the k-Riemann-Liouville fractional derivative is given by*

$$\mathfrak{L}\left[D_k^{\beta}f\right](s) = s.\,(ks)^{-\frac{1-\beta}{k}}\,\mathfrak{L}\left[f\right](s) - I_k^{1-\beta}f(0) \tag{II.16}$$

Proof.

By definition (II.1) and property (II.14) we arrive at

$$\mathfrak{L}\left[D_k^{\beta}f(t)\right](s) = \mathfrak{L}\left[\frac{d}{dt}I_k^{1-\beta}f(t)\right](s) =$$

$$s\mathfrak{L}\left[I_k^{1-\beta}f(t)\right](s) - I_k^{1-\beta}f(0) \tag{II.17}$$

From (II.8) and (II.17) we have

$$\mathfrak{L}\left[D_k^{\beta}f\right](s) = s.\,(ks)^{-\frac{1-\beta}{k}}\,\mathfrak{L}\left[f\right](s) - I_k^{1-\beta}f(0)$$

Lemma 4 *Let β be a real number, $0 < \beta \leq 1$. The Fourier transform of the k-Riemann-Liouville fractional derivative is given by*

$$\mathfrak{F}\left[D_k^{\beta}f\right](\omega) = (-i\omega).\,(-i\omega k)^{-\frac{1-\beta}{k}}\,\mathfrak{F}\left[f\right](\omega) \tag{II.18}$$

Proof.

By definition (II.1) and property (II.15) we have

$$\mathfrak{F}\left[D_k^{\beta}f(t)\right](\omega) = \mathfrak{F}\left[\frac{d}{dt}I_k^{1-\beta}f(t)\right](\omega) =$$

$$(-i\omega)\mathfrak{F}\left[I_k^{1-\beta}f(t)\right](\omega) \tag{II.19}$$

From (II.13) and (II.19) we have

$$\mathfrak{F}\left[D_k^{\beta}f\right](\omega) = (-i\omega).\,(-i\omega k)^{-\frac{1-\beta}{k}}\,\mathfrak{F}\left[f\right](\omega)$$

Examples of application

Example 1

$$\left(D_{k,a}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{k^{1-n}\Gamma_k(k\beta)}{\Gamma_k\left(n-\alpha+k\beta-kn\right)}\,(x-a)^{\frac{n-\alpha}{k}+\beta-(n+1)}$$

Proof.

$$\left(D_{k,a}^{\alpha}(t-a)^{\beta-1}\right)(x) = \left(\frac{d}{dx}\right)^{n}\left(I_{k,a}^{n-\alpha}(t-a)^{\beta-1}\right)(x) \tag{II.20}$$

In (II.20), calling $p = n-\alpha$ we calculate

$$\left(I_{k,a}^{n-\alpha}(t-a)^{\beta-1}\right)(x) = \left(I_{k,a}^{p}(t-a)^{\beta-1}\right)(x) =$$

$$\frac{1}{k\Gamma_k(p)}\int_a^x (x-t)^{\frac{p}{k}-1}(t-a)^{\beta-1}dt \tag{II.21}$$

Making the change of variables $t = a+\varepsilon(x-a)$ it result

$$\left(I_{k,a}^{n-\alpha}(t-a)^{\beta-1}\right)(x) = \frac{1}{k\Gamma_k(p)}\int_0^1 \left[(1-\varepsilon)((x-a))\right]^{\frac{p}{k}-1}\left[\varepsilon(x-a)\right]^{\beta-1}(x-a)d\varepsilon =$$

$$\frac{1}{\Gamma_k(p)}(x-a)^{\frac{p}{k}+\beta-1}\frac{1}{k}\int_0^1 (1-\varepsilon)^{\frac{p}{k}-1}\varepsilon^{\beta-1}d\varepsilon =$$

$$\frac{(x-a)^{\frac{p}{k}+\beta-1}}{\Gamma_k(p)}B_k(p,k\beta) \tag{II.22}$$

where $B_k(x,y)$ denote the k-Beta function.
Remembering that $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$, it result

$$\left(I_{k,a}^{n-\alpha}(t-a)^{\beta-1}\right)(x) = \frac{(x-a)^{\frac{p}{k}+\beta-1}}{\Gamma_k(p)} \frac{\Gamma_k(p)\Gamma_k(k\beta)}{\Gamma_k(p+k\beta)} =$$

$$\frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)}(x-a)^{\frac{n-\alpha}{k}+\beta-1} \tag{II.23}$$

Replacing (II.23) in (II.20) we have

$$\left(D_{k,a}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)}\left(\frac{d}{dx}\right)^n (x-a)^{\frac{n-\alpha}{k}+\beta-1} =$$

$$\frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta)} \frac{\Gamma_k(n-\alpha+k\beta)}{\Gamma_k(n-\alpha+k\beta-kn)} \frac{k^{1-\frac{n-\alpha+k\beta}{k}}}{k^{1-\frac{n-\alpha+k\beta-kn}{k}}}(x-a)^{\frac{n-\alpha}{k}+\beta-(n+1)} =$$

$$\frac{\Gamma_k(k\beta)}{\Gamma_k(n-\alpha+k\beta-kn)}k^{1-n}(x-a)^{\frac{n-\alpha}{k}+\beta-(n+1)} \tag{II.24}$$

Example 2

$$\left(D_{k,a}^{\alpha}1\right)(x) = \frac{k^{1-n}(x-a)^{\frac{n-\alpha}{k}-n}}{\Gamma_k(n-\alpha+k(1-n))}$$

Proof.

It is sufficient take $\beta = 1$ in Example 1.

Example 3

$$\left(D_{k,a}^{\alpha}(t-a)^{\alpha-j}\right)(x) = \frac{k^{1-n}\Gamma_k(n\alpha-kj+k)}{\Gamma_k\left(n-\alpha+k\alpha-jk+k-kn\right)}(x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}}$$

Proof.

$$\left(D_{k,a}^{\alpha}(t-a)^{\alpha-j}\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{k,a}^{n-\alpha}(t-a)^{\alpha-j}\right)(x) \tag{II.25}$$

In (II.25), calling $p = n-\alpha$ we calculate

$$\left(I_{k,a}^{p}(t-a)^{\alpha-j}\right)(x) = \frac{1}{k\Gamma_k(p)}\int_a^x (x-t)^{\frac{p}{k}-1}(t-a)^{\alpha-j}dt \tag{II.26}$$

Making the change of variables $t = a + \varepsilon(x - a)$ it result

$$\left(I^{p}_{k,a}(t-a)^{\alpha-j}\right)(x) = \frac{1}{k\Gamma_k(p)}\int_0^1 \left[(1-\varepsilon)((x-a))\right]^{\frac{p}{k}-1}\left[\varepsilon(x-a)\right]^{\alpha-j}(x-a)d\varepsilon =$$

$$\frac{1}{\Gamma_k(p)}(x-a)^{\frac{p}{k}+\alpha-j}\frac{1}{k}\int_0^1 (1-\varepsilon)^{\frac{p}{k}-1}\varepsilon^{\frac{k(\alpha-j+1)}{k}-1}d\varepsilon =$$

$$\frac{(x-a)^{\frac{p}{k}+\alpha-j}}{\Gamma_k(p)}B_k(p,k(\alpha-j+1))$$

$$\frac{(x-a)^{\frac{p}{k}+\alpha-j}}{\Gamma_k(p)}\frac{\Gamma_k(p)\Gamma_k(k(\alpha-j+1))}{\Gamma_k(p+k(\alpha-j+1))} =$$

$$\frac{\Gamma_k(k(\alpha-j+1))}{\Gamma_k(n-\alpha+k(\alpha-j+1))}(x-a)^{\frac{n-\alpha}{k}+\alpha-j} \tag{II.27}$$

Replacing (II.27) in (II.25) we have

$$\left(D^{\alpha}_{k,a}(t-a)^{\alpha-j}\right)(x) = \frac{\Gamma_k(k(\alpha-j+1))}{\Gamma_k(n-\alpha+k(\alpha-j+1))}\left(\frac{d}{dx}\right)^n (x-a)^{\frac{n-\alpha}{k}+\alpha-j} =$$

$$\frac{\Gamma_k(n\alpha-kj+k)}{\Gamma_k(n-\alpha+k\alpha-jk+k-kn)}k^{1-n}(x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}} \tag{II.28}$$

Example 4

$$D^{\alpha}_{k,a}\left[\sum_{j=1}^{n} c_j(x-a)^{\alpha-j}\right](x) =$$

$$\sum_{j=1}^{n} c_j\left(\frac{\Gamma_k(n\alpha-kj+k)}{\Gamma_k(n-\alpha+k\alpha-jk+k-kn)}k^{1-n}(x-a)^{\frac{n-\alpha+k\alpha-jk-kn}{k}}\right)$$

Proof.
From (II.28) we have the example 4.

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