A Laplace Type Problem for an Irregular Lattice and "Body Test" Rectangle

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Abstract
In this paper we want to compute the probability that the rectangle of constant dimension and random position intersects a side of the lattice with the fundamental cell represented in the figure 1.

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Let $\mathcal{R}(a, \alpha, \beta)$ the lattice with fundamental cell $C_0 = C_{01} \cup C_{02}$ represented in the figure 1 where $\alpha$ and $\beta$ are angles with $\frac{\pi}{4} < \alpha \leq \frac{\pi}{3}$ and $\beta \leq \alpha$. We have

$$|AB| = |EF| = \frac{a}{2\cos\alpha},$$

$$|BC| = |DE| = \frac{a}{2\cos\beta},$$

(1)

and
We want to compute the probability that a rectangle $r$ intersects a side of the lattice $R$ therefore the probability $P_{int}$ that $r$ intersects a side of the fundamental cell $C_0$.

The position of the rectangle $r$ is determined by its center of the rectangle and the angle $\varphi$ represented in the figure 2.

To compute the probability $P_{int}$ we consider the limit positions of the rectangle $r$, for a fixed value of $\varphi$, situated in the cell $C_{01}$. So we have the figure

\[
\text{area}C_{01} = \frac{3a^2}{4}\tan \alpha, \quad \text{area}C_{02} = \frac{3a^2}{4}\tan \beta. \quad (2)
\]
and the formulas

$$\text{area} \hat{C}_{01}(\varphi) = \text{area} C_{01} - [\text{area} a_1(\varphi) + \text{area} a_2(\varphi) + \ldots + \text{area} a_{12}(\varphi)] \quad (3)$$

and

$$\text{area} \hat{C}_{02}(\varphi) = \text{area} C_{02} - [\text{area} b_1(\varphi) + \text{area} b_2(\varphi) + \ldots + \text{area} a_{12}(\varphi)] \quad (4)$$

We consider the figure
From here follow

\[ \overline{A_1 AA_4} = \pi - \varphi, \quad \overline{AA_1 A_4} = \alpha - \varphi, \]  

therefore

\[ \text{area}a_1 (\varphi) = \frac{l^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha}. \]  

The figure

\[ \overline{BB_4 B_1} = \frac{\pi}{2} - \varphi, \quad \overline{BB_1 B_4} = \frac{\pi}{2} - \alpha + \varphi \]  

and

\[ |BB_1| = \frac{m \cos \varphi}{\sin \alpha}, \quad |BB_4| = \frac{m \cos (\alpha - \varphi)}{\sin \alpha}. \]  

Therefore

\[ \text{area}a_4 (\varphi) = \frac{m^2 \cos \varphi \cos (\alpha - \varphi)}{2 \sin \alpha}. \]  

From the figure
we obtain
\[ \widehat{B_2B_1B_6} = \widehat{AA_1A_4} = \alpha - \varphi, \]
\[ \widehat{B_1B_6B_2} = \frac{\pi}{2} - \alpha + \varphi \] (10)

and
\[ |B_5B_6| = l \tan (\alpha - \varphi), \quad |B_1B_6| = \frac{l}{\cos (\alpha - \varphi)}, \] (11)

therefore
\[ \text{area}_a3(\varphi) = \frac{l^2}{2} \tan (\alpha - \varphi). \] (12)

From the figure 2 we obtain
\[ \Delta O_1A_2A_7 = \Delta O_2B_2B_7, \]
\[ |A_2A_7| = |B_2B_7|, \quad |A_1A_7| = |B_6B_7|. \]

From here and with the first relation (11) we have
\[ |A_2A_7| = \frac{1}{2} \left[ m - l \tan (\alpha - \varphi) \right], \]
\[ |A_1A_7| = \frac{1}{2} \left[ m + l \tan (\alpha - \varphi) \right]. \]

Now we consider the figure
Considering of the second expression (10) we have

\[ h_2 = \frac{1}{2} [l \sin (\alpha - \varphi) + m \cos (\alpha - \varphi)] . \]

Moreover the formulas (1), (5), (8) and (11) give us

\[
|A_1B_6| = \frac{a}{2 \cos \alpha} - |AA_1| - |B_1B_6| - |BB_1| =
\]

\[
\frac{a}{2 \cos \alpha} - \frac{l \sin \varphi + m \cos \varphi}{\sin \alpha} - \frac{l}{\cos (\alpha - \varphi)}. \]

From the last two relations follow

\[
\text{area}_{a_2}(\varphi) = \frac{1}{2} \left( \frac{a}{2 \cos \alpha} - \right.
\]

\[
\frac{l \sin \varphi + m \cos \varphi}{\sin \alpha} \left[ l \sin (\alpha - \varphi) +
\right.
\]

\[
\frac{m \cos (\alpha - \beta)}{2} - \frac{lm}{2} - \frac{l^2}{2} \tan (\alpha - \varphi). \tag{13}
\]

The formulas (6), (9), (12) and (13) give us

\[
\text{area}_{a_1}(\varphi) + \text{area}_{a_2}(\varphi) + \text{area}_{a_3}(\varphi) + \text{area}_{a_4}(\varphi) =
\]

\[
\frac{a}{4 \cos \alpha} [l \sin (\alpha - \varphi) + m \cos (\alpha - \varphi)] - lm. \tag{14}
\]
Laplace type problem

\[
\frac{E_3 E_4 E_8}{\pi} = \frac{\pi}{2} - \varphi
\]
and
\[
|E_3 E_8| = m \cot \varphi, \quad |E_4 E_8| = \frac{m}{\sin \varphi},
\]
therefore
\[
\text{area}_{a_6}(\varphi) = \frac{m^2}{2} \cot \varphi.
\]

Then from the figure

\[
\frac{E_3 E_4}{\pi} = \pi - (\alpha + \varphi)
\]
and
\[
|EE_1| = \frac{l \sin \varphi}{\sin \alpha}, \quad |EE_4| = \frac{l \sin (\alpha + \varphi)}{\sin \alpha}.
\]
Therefore
\[
\text{area}_{a_7}(\varphi) = \frac{l^2 \sin \varphi \sin (\alpha + \varphi)}{2 \sin \alpha}.
\]

The figure 2 give us
\[
|B_3 B_8| = |E_3 E_5|, \quad |B_4 B_8| = |E_5 E_8|.
\]
Considering here of the first relation (15) we obtain
\[
|B_4 B_8| = \frac{1}{2} (l + m \cot \varphi).
\]

We consider the figure
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we have

\[ h_5 = \frac{1}{2} (l \sin \varphi + m \cos \varphi). \]

In the same way with the formulas (8), (15) and (18) we can write

\[ |B_4E_8| = 2a - |BB_4| - |E_4E_8| - |EE_4| = \]
\[ 2a - \frac{l \sin (\alpha + \varphi) + m \cos (\alpha - \varphi)}{\sin \alpha} - \frac{m}{\sin \varphi}. \]

Consequently we have

\[ \text{area}_{a5}(\varphi) = a(l \sin \varphi + m \cos \varphi) - \]
\[ \frac{(l \sin \varphi + m \cos \varphi) [l \sin (\alpha + \varphi) + m \cos (\alpha - \varphi)]}{2 \sin \alpha} - \frac{lm}{2} - \frac{m^2}{2 \cot \varphi}. \]

This relation and the (16) give us

\[ \text{area}_{a5}(\varphi) + \text{area}_{a6}(\varphi) = a(l \sin \varphi + m \cos \varphi) - \]
\[ \frac{lm}{2 \cot \alpha \cdot \sin 2\varphi} - \frac{m^2 \cos \varphi \cos (\alpha - \varphi)}{2 \sin \alpha} - lm. \quad (20) \]

Now we consider the figure

\[ \text{fig.10} \]
Laplace type problem

Considering the formula (17) we have

\[ \widehat{E_1E_7E_2} = \widehat{EE_1E_4} = \pi - (\alpha + \varphi), \]
\[ \widehat{E_2E_1E_7} = \alpha + \varphi - \frac{\pi}{2} \]  

(21)

and

\[ |E_2E_7| = -m \cot(\alpha + \varphi), \]
\[ |E_1E_7| = \frac{m}{\sin(\alpha + \varphi)}. \]  

(22)

Therefore

\[ \text{area}_{a8}(\varphi) = \frac{-m^2}{2} \cot(\alpha + \varphi). \]  

(23)

From the figure

\[ \widehat{FF_1F_4} = \pi - \alpha, \quad \widehat{F_4F_1} = \frac{\pi}{2} - \varphi, \]
\[ \widehat{FF_1F_4} = \alpha + \varphi - \frac{\pi}{2} \]  

(24)

and

\[ |FF_1| = \frac{m \cos \varphi}{\sin \alpha}, \quad |FF_4| = -\frac{m \cos (\alpha + \varphi)}{\sin \alpha}, \]  

(25)

therefore

\[ \text{area}_{a10}(\varphi) = -\frac{m^2 \cos \varphi \cos (\alpha + \varphi)}{2 \sin \alpha}. \]  

(26)

From figure 2 and from formula (22) we obtain

\[ |E_6E_7| = \frac{1}{2} |l - m \cot(\alpha + \varphi)| \]

We consider the figure
Considering of the formula (17) we have

\[ \widehat{E_7F_1F_6} = \widehat{EE_1E_4} = \pi - (\alpha + \varphi), \]

therefore

\[ h_9 = \frac{1}{2} \left[ l \sin (\alpha + \varphi) - m \cos (\alpha + \varphi) \right]. \]

Moreover, with the formulas (18), (22) and (25) we can write

\[ |E_7E_1| = \frac{a}{2 \cos \alpha} - |EE_1| - |E_1E_7| - |FF_1| = \]

\[ \frac{a}{2 \cos \alpha} - \frac{l \sin \varphi + m \cos \varphi}{\sin \alpha} - \frac{m}{\sin (\alpha + \varphi)}. \]

Hence

\[ \text{area}_{a9} (\varphi) = \frac{a}{4 \cos \alpha} \left[ l \sin (\alpha + \varphi) - m (\alpha + \varphi) \right] - \]

\[ \frac{l^2 \sin \varphi \sin (\alpha + \varphi) - m^2 \cos \varphi \cos (\alpha + \varphi)}{2 \sin \alpha} \]

\[ l m + \frac{m^2}{2} \cot (\alpha + \varphi). \] (27)

From the formulas (23), (26) and (27) follow

\[ \text{area}_{a8} + \text{area}_{a9} (\varphi) + \text{area}_{a10} (\varphi) = \]

\[ \frac{a}{4 \cos \alpha} \left[ l \sin (\alpha + \varphi) - m \cos (\alpha + \varphi) \right] - \]
Now we consider the figure

![Diagram of figure 13](image)

We have $A_3A_4A_5 = \frac{\pi}{2} - \varphi$ and

$$|A_3A_5| = m \cot \varphi, \quad |A_4A_5| = \frac{m}{\sin \varphi},$$

therefore

$$\text{area}_{A_1A_2} (\varphi) = \frac{m^2}{2} \cot \varphi.$$  \hspace{1cm} (30)

from the figure 2 we have

$$|A_3A_6| = |F_3F_5|,$$

$$|A_5A_6| = |F_4F_5|.$$  

from here and with the first relation (29) we obtain

$$|A_5A_6| = \frac{1}{2} (l + m \cot \varphi).$$

The figure

![Diagram of figure 14](image)
give us

\[ h_{11} = \frac{1}{2} (l \sin \varphi + m \cos \varphi). \]

At the end, from the formulas (5), (25) and (29) we have

\[ |A_5F_4| = a - |AA_4| - |A_4A_5| - |FF_4| = \]
\[ a - \frac{l \sin (\alpha - \varphi) - m \cos (\alpha + \varphi)}{\sin \alpha} - \frac{m}{\sin \varphi}. \]

Therefore

\[ area_{a11} (\varphi) = \frac{a}{2} (l \sin \varphi + m \cos \varphi) - \]
\[ \frac{(l \sin \varphi + m \cos \varphi) [l \sin (\alpha - \varphi) - m \cos (\alpha + \varphi)]}{2 \sin \alpha} - \frac{lm}{2} - \frac{m^2}{2} \cot \varphi. \]  

(31)

This formula and the formula (31) give us

\[ area_{a11} + area_{a12} (\varphi) = \frac{a}{2} (l \sin \varphi - m \cos \varphi) - \]
\[ \frac{l^2 - m^2 \cot \alpha}{4} \cos 2\varphi - \frac{l^2 + m^2 - 2lm \cot \alpha}{4} \sin 2\varphi + \]
\[ \frac{l^2 + m^2}{4} \cot \alpha - lm. \]  

(32)

Replacing in the (3) the expressions (14), (19), (20), (28) and (31) we obtain

\[ area_{\hat{C}01} (\varphi) = areaC_{01} - \left\{ \frac{a}{2} [(3l + m \tan \alpha) \sin \varphi + \]
\[ (3m + l \tan \alpha) \cos \varphi] - \frac{l^2 + m^2}{4} \sin 2\varphi - \]
\[ \frac{l^2}{4} \cos 2\varphi + \frac{l^2 + m^2}{4} \cot \alpha \right\}. \]  

(33)

Replancing \( \alpha \) with \( \beta \) we can write

\[ area_{\hat{C}02} (\varphi) = areaC_{02} - \left\{ \frac{a}{2} [(3l + m \tan \beta) \sin \varphi + \]
\[ (3m + l \tan \beta) \cos \varphi] - \frac{l^2 + m^2}{4} \sin 2\varphi - \]
\[ \frac{l^2}{4} \cos 2\varphi + \frac{l^2 + m^2}{4} \cot \beta \right\}. \]
Laplace type problem

\[ \frac{l^2}{4} \cos 2\varphi + \frac{l^2 + m^2}{4} \cot \beta \]. \quad (34)

Denoting with \( M_1 \), the set of the rectangles \( r \) that have the centre in the cell \( C_{01} \) and with \( N_1 \) the set of the rectangles completely contained in \( C_{01} \). We have

\[ \varphi \epsilon [0, \alpha] . \]

Therefore denoting with \( \mu \) the Lebesgue measure in Euclidean plane and the measures Poincaré kinematic measure [2]:

\[ dK = d\varphi \wedge dx \wedge dy, \]

where \( x, y \) are the coordinate of the centre rectangles \( r \) and \( \varphi \) is the angle already defined, we have

\[ \mu (M_1) = \int_0^\alpha d\varphi \int\int_{\{(x,y) \in C_{01}\}} dx dy = \]

\[ \int_0^\alpha (\text{area}C_{01}) d\varphi = \alpha \text{area}C_{01} \]

and considering of the formula (32),

\[ \mu (N_1) = \int_0^\alpha d\varphi \int\int_{\{(x,y) \in \tilde{C}_{01}(\varphi)\}} dx dy = \]

\[ \int_0^\alpha \left[ \text{area}\tilde{C}_{01} (\varphi) \right] d\varphi = \alpha \text{area}C_{01} - \]

\[ \frac{\alpha}{2} \left[ \left( 3 - 4 \cos \alpha + \frac{1}{\cos \alpha} \right) l - \sin \alpha \left( 2 + \frac{1}{\cos \alpha} \right) \right] m + \]

\[ \frac{l^2}{4} \left[ \sin \alpha (\sin \alpha + \cos \alpha) - \alpha \cot \alpha \right] + \]

\[ \frac{m^2}{4} \left( \sin^2 \alpha - \alpha \cot \alpha \right) . \]

(36)

In the same way, using the formula (33), we have

\[ \mu (M_2) = \int_0^\beta d\varphi \int\int_{\{(x,y) \in C_{02}\}} dx dy = \]
\[
\int_0^\beta (\text{area} C_{02}) \, d\varphi = \beta \text{area} C_{02},
\]

(37)

\[
\mu (N_2) = \int_0^\beta d\varphi \iint \{ (x, y) \in \hat{C}_{02}(\varphi) \} \, dx \, dy = \int_0^\beta \left[ \text{area} \hat{C}_{02}(\varphi) \right] d\varphi = \beta \text{area} C_{02} -
\]

\[
\frac{\alpha}{2} \left[ \left( 3 - 4 \cos \beta + \frac{1}{\cos \beta} \right) l - \sin \beta \left( 2 + \frac{1}{\cos \beta} \right) m \right] + \frac{l^2}{4} \left[ \sin \beta (\sin \beta + \cos \beta) - \beta \cot \beta \right] + \frac{m^2}{4} \left( \sin^2 \beta - \beta \cot \beta \right).
\]

(38)

With these definitions we have [3]:

\[
P_{\text{int}} = 1 - \frac{\mu (N_1) + \mu (N_2)}{\mu (M_1) + \mu (M_2)}.
\]

(39)

The formulas (34), (35), (36), (37) and (38) give us

\[
P_{\text{int}} = \frac{1}{3a^2 (\alpha \tan \alpha + \beta \tan \beta)} \left\{ a \left[ (6 - 4 \cos \alpha -
\right.ight.
\]

\[
\left. 4 \cos \beta + \frac{1}{\cos \alpha} + \frac{1}{\cos \beta} \right] l - (2 \sin \alpha +
\]

\[
2 \sin \beta + \tan \alpha + \tan \beta \right) m \right] - \frac{l^2}{2} \left[ \sin \alpha (\sin \alpha + \cos \alpha) + \sin \beta (\sin \beta + \cos \beta) - \alpha \cot \alpha - \beta \cot \beta \right] - \frac{m^2}{2} \left( \sin^2 \alpha + \sin^2 \beta - \alpha \cot \alpha - \beta \cot \beta \right) \right\}.
\]

For \( m = 0 \) the rectangle \( r \) became a segment of length \( l \) and we find

the probability determined in [1].
References


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