Fixed Point Theorems for Random Operators
in Hilbert Space

Rajesh Shrivastava*, Anil Rajput** and Shiv Kumar Singh***

*Govt. Science & Commerce Benazeer College, Bhopal, India rajeshraju0101@rediffmail.com
**Bhabha Engineering Research Institute – MCA, Bhopal, India dranilrajput@hotmail.com
***Sagar Institute of Science and Technology, Bhopal, India lekh13@yahoo.com

Abstract
Our main aim of this paper is introduced some new unique common random fixed point theorems of random operators in Hilbert Space by considering a sequence of measurable functions satisfying conditions A or B and C. Our results are motivated from [3, 5, 6, 7, 8].

Mathematics Subject Classification: 54H25, 47H10.

Keywords: Separable Hilbert space, Random operators, Common random fixed point, Cauchy sequence
1. Introduction and preliminaries

In recent years, the study of random fixed points has attracted much attention; some of the recent literatures in random fixed point may be noted in [1, 2, 3]. In this paper we construct a sequence of measurable function and consider its convergence to the common unique random fixed point of two continuous random operators defined on a non-empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the two continuous random operators. We have introduced a rational inequality and used the parallelogram law. Throughout this paper, \((\Omega, \mathcal{F})\) denotes a measurable space consisting of a set \(\Omega\) and sigma algebra \(\mathcal{F}\) of subset of \(\Omega\). \(H\) stands for a separable Hilbert space and \(C\) is nonempty closed subset of \(H\).

1.1 Definition: A function \(f: \Omega \rightarrow \mathcal{C}\) is said to be measurable if \(f^{-1}(B \cap C) \in \mathcal{F}\) for every Borel subset \(B\) of \(H\).

1.2. Definition: A function \(F: \Omega \times \mathcal{C} \rightarrow \mathcal{C}\) is said to be a random operator if \(F(. , x) : \Omega \rightarrow \mathcal{C}\) is measurable for every \(x \in \mathcal{C}\).

1.3. Definition: A measurable function \(g: \Omega \rightarrow \mathcal{C}\) is said to be a random fixed point of the random operator \(F: \Omega \times \mathcal{C} \rightarrow \mathcal{C}\) if \(F(t(g(t))) = g(t)\) for all \(t \in \Omega\).

1.4. Definition: A random operator \(F: \Omega \times \mathcal{C} \rightarrow \mathcal{C}\) is said to be continuous if for fixed \(t \in \Omega\), \(F(t, .) : \mathcal{C} \rightarrow \mathcal{C}\) is continuous.

Condition A: Let \(S, T: \mathcal{C} \rightarrow \mathcal{C}\) be three mapping, when \(C\) is non empty subset of a Hilbert space \(H\), is said to satisfy the condition A, if

\[
K_{\text{max}} \leq \frac{\|x - F^s x\|^2}{2}, \quad \frac{\|x - E^p x\|^2}{2} + \frac{\|y - F^s y\|^2}{2} + \alpha \|x - y\|^2, \quad 0 \leq K \leq 1/2; \quad c \geq 0 \text{ and } r, s > 0
\]

Condition B: At each pair of points \(x, y\) mapping \(E, F \& C\) satisfies at least one of the following conditions

\[
\|x - E^p x\|^2 + \|y - F^s y\|^2 \leq \alpha \|x - y\|^2, \quad 1 \leq \alpha \leq 2.
\]
Fixed point theorems for random operators

Condition C: Two mapping $S, T : C \rightarrow C$, where $C$ is a non-empty closed subset of a Hilbert space $H$, is said to satisfy condition (A) if

$$\|S_x - T_y\| \leq \frac{1}{1 + \alpha} \left( |1 + \|x - S_x\|^2| \right) + b \left( |\|x - S_x\|^2 + \|y - T_y\|^2| \right) + C \|S_x - T_y\| \big]$$

For each $x, y$ in $C$, $a, b$, being positive real number such that $0 < a + b < 1/2$

2. Main Results

2.1. Theorem: Let $C$ be a nonempty subset of Hilbert Space $H$. Let $S$ and $T$ be three continuous random operations defined on $C$ such that for $x, y \in C$ satisfying condition A or B. Then the sequence $(S_n)$ defined in (1) converges to the unique common random fixed point of $E^r$ and $F^s$.

Proof A: $(S_n)$ is sequence of function defined in (1). For $\xi \in C$ and $n = 0, 1, 2, 3, \ldots$

$$\|S_{2n+1}(\xi) - S_{2n}(\xi)\|^2 = \|E^r (S_{2n+1}(\xi)) - E^r (S_{2n}(\xi))\|^2$$

$\leq \text{Kmax} \left[ c \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 \left( \|S_{2n}(\xi) - E^r (S_{2n}(\xi))\|^2 + \|S_{2n}(\xi) - E^r (S_{2n-1}(\xi))\|^2 \right) \right]

\[ \frac{1}{2} \left[ \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 + \|S_{2n}(\xi) - E^r (S_{2n-1}(\xi))\|^2 \right] \]

$= \text{Kmax} \left[ c \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2, \|S_{2n}(\xi) - S_{2n+1}(\xi)\|^2 \right]

\[ \frac{1}{2} \left[ \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 + \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 \right] \]

$= \text{Kmax} \left[ c \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2, \|S_{2n}(\xi) - S_{2n+1}(\xi)\|^2 \right]

\[ \frac{1}{2} \left[ \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 + \|S_{2n}(\xi) - S_{2n+1}(\xi)\|^2 \right] \]

$= \text{Kmax} \left[ c \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2, \|S_{2n}(\xi) - S_{2n+1}(\xi)\|^2 \right]

\[ \frac{1}{2} \left[ \|S_{2n}(\xi) - S_{2n-1}(\xi)\|^2 + \|S_{2n}(\xi) - S_{2n+1}(\xi)\|^2 \right] \]

(by parallelogram law)
\[ \|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 \leq k \|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 \]

Also, \[ \|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 \leq \frac{k}{1+n} \|g_{2n-2}(\xi) - g_{2n-1}(\xi)\|^2 \]

Hence in general \[ \|g_n(\xi) - g_{n+1}(\xi)\|^2 \leq \frac{k}{1+n} \|g_{n-1}(\xi) - g_n(\xi)\|^2 \]

Since \(0 < \frac{k}{1+n} < 1\), therefore \(\{g_n(\xi)\}\) be Cauchy sequence and hence convergent in H.

Therefore \(g_n(\xi) \to g(\xi)\) as \(n \to \infty\). Since C is closed, be a function. For all

\[ g(\xi) = \Omega \rightarrow C \text{ is a continuous random operation on a non empty subset of a separated Hilbert space } H, \text{ then any measurable formula } f: \Omega \rightarrow C \text{ the function } g(\xi) = G(f(\xi)) \text{ is also measurable. Therefore the sequence of measurable function } \{g_n\} \text{ converges to measurable function of this fact along with } E^f(\xi, g(\xi)) = g(\xi) \]

Now, if \( G: \Omega \times C \to C \) is a continuous random operation on a non empty subset of a separated Hilbert space H, then any measurable formula \( f: \Omega \to C \) the function \( g(\xi) = G(f(\xi)) \) is also measurable. Therefore the sequence of measurable function \( \{g_n\} \) converges to measurable function of this fact along with \( E^f(\xi, g(\xi)) = g(\xi) \) shows that \( g: \Omega \times C \to C \) common random fixed point of \( F^1 \) and \( F^2 \).
Fixed point theorems for random operators

Uniqueness:

Let h: \( \Omega \rightarrow C \) be another common random fixed point of \( E^r \) and \( F^s \).

\[
\| \gamma(\xi) - h(\xi) \|^2 = \| E^r (\xi, \gamma(\xi)) - F^s (\xi, h(\xi)) \|^2 \\
\text{kmax}[c \| \gamma(\xi) - h(\xi) \|^2, \| \gamma(\xi) - E^r (\xi, \gamma(\xi)) \|^2, \| h(\xi) - F^s (\xi, h(\xi)) \|^2] \\
= \text{kmax}[c \| \gamma(\xi) - h(\xi) \|^2, \| \gamma(\xi) - E^r (\xi, \gamma(\xi)) \|^2, \| h(\xi) - F^s (\xi, h(\xi)) \|^2] \\
= k \| \gamma(\xi) - h(\xi) \|^2 \\
\gamma(\xi) = h(\xi) \ \forall \ \xi \in \Omega \quad \text{(as } k < 1/2)\
\]

**Proof B:** Suppose sequence \( \{\gamma_n\} \) satisfy (a) of condition B then

\[
\| \gamma_{2n+1}(\xi) - F^s (\xi, \gamma_{2n+1}(\xi)) \|^2 + \| \gamma_{2n+2}(\xi) - F^s (\xi, \gamma_{2n+2}(\xi)) \|^2 \\
\leq \alpha \| \gamma_{2n}(\xi) - \gamma_{2n-1}(\xi) \|^2 \\
\| \gamma_{2n+1}(\xi) + \gamma_{2n+2}(\xi) \|^2 + \| \gamma_{2n+1}(\xi) - \gamma_{2n+2}(\xi) \|^2 \\
\leq \| \gamma_{2n}(\xi) + \gamma_{2n+1}(\xi) \|^2 \\
\| \gamma_{2n+1}(\xi) - \gamma_{2n+2}(\xi) \|^2 \leq (\alpha - 1) \| \gamma_{2n}(\xi) + \gamma_{2n+1}(\xi) \|^2 \\
\]

Similarly by (b), (c) we get

\[
\| \gamma_{2n+1}(\xi) - \gamma_{2n+2}(\xi) \|^2 \leq \frac{2\alpha - 1}{2 - \alpha} \| \gamma_{2n}(\xi) - \gamma_{2n+1}(\xi) \|^2 \\
& \| \gamma_{2n+1}(\xi) - \gamma_{2n+2}(\xi) \|^2 \leq \frac{2\alpha - 1}{2 - \alpha} \| \gamma_{2n}(\xi) - \gamma_{2n+1}(\xi) \|^2 \\
\]

By above, we have

\[
\| \gamma_{2n+1}(\xi) - \gamma_{2n+2}(\xi) \|^2 \leq \lambda \| \gamma_{2n}(\xi) - \gamma_{2n+1}(\xi) \|^2 \\
\]

Where \( \lambda = \max \{(\alpha - 1), \frac{2\alpha - 1}{2 - \alpha}, \frac{2\alpha - 1}{2 - \alpha}\} < 1 \).

Repeating the same process as in **proof (A)**, we find \( \gamma(\xi) \) is unique common random fixed point of \( E^r \) and \( F^s \) for all \( r, s > 0 \).

**Examples:**

Let \( \Omega = [0, 1] \) and \( \Sigma \) be the sigma algebra of Lebesgue’s measurable subset of \([0, 1]\). Let \( H = \mathbb{R} \) with \( ||x - y|| = |a^{x-y} - 1| \), where \( a > 1 \). Define random operators \( E^r \) and \( F^s : \Omega \times X \rightarrow H \) as \( E^r (\xi, x) = (1-x^2+2x)/3 \) and
\( F^x (\xi, x) = (1 - \xi^2 + 3x)/4. \) Also sequence of mapping \( g_n : \Omega \to X \) is defined by \( g_n(x) = 1 + (1/n) - F^x \) for every \( x \in \Omega \) and \( n. \) Define measurable mapping \( g : \xi, \Omega \to X \) as \( g(\xi) = 1 - F^x \) for every \( \xi \in \Omega. \)

\[
\lim_{n \to \infty} \left\| E^n \left( g_n(\xi) - g(\xi) \right) \right\| = \lim_{n \to \infty} \left| E^n \left( g_n(\xi) - g(\xi) \right) \right| = 0
\]

Clearly \( E^n \) and \( F^x \) satisfy conditions A and B.

2.2. Theorem: Let \( C \) be a non-empty closed subset of a separable Hilbert space \( H. \) Let \( S \) and \( T \) be two continuous random operators defined on \( C \) such that for \( t \in \Omega, S(t, \cdot), T(t, \cdot) : C \) satisfy condition \( (C). \) Then \( S \) and \( T \) have a common unique random fixed point in \( C. \)

Proof C: We define a sequence of function \( \{ g_n \} \) as \( g_n : \Omega \to C \) as arbitrary measurable function for \( t \in \Omega \) and \( n = 0, 1, 2, 3 \ldots. \)

\[
\begin{align*}
g_{2n+1}(t) &= S(t, g_{2n}(t)), \\ g_{2n+2}(t) &= T(t, g_{2n+1}(t))
\end{align*}
\]

(1)

If \( g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t) \) for \( t \in \Omega, \) for some \( n \) then we see that \( g_{2n}(t) \) is a random fixed point of \( S \) and \( T. \) Therefore we suppose that no two consecutive terms of sequence \( \{ g_n \} \) are equal. Now consider for \( t \in \Omega \)

\[
\begin{align*}
\left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 &= \left\| S(t, g_{2n}(t)) - T(t, g_{2n+1}(t)) \right\|^2 \\
&\leq \left( \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \\
&\leq \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + b\left( \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \\
&\leq \left( \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + b\left( \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \right) \\
&= \left( \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + b\left( \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \right) \\
&\leq \left( \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + b\left( \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \right) \\
&= \left( \left\| g_{2n+1}(t) - g_{2n+2}(t) \right\|^2 + b\left( \left\| g_{2n}(t) - S(t, g_{2n}(t)) \right\|^2 + \left\| g_{2n+1}(t) - T(t, g_{2n+1}(t)) \right\|^2 \right) \right)
\end{align*}
\]
Fixed point theorems for random operators

\[ \Rightarrow \| (g_{2n+1}(t) - g_{2n+2}(t)) \| \leq K \| g_{2n}(t) - g_{2n+1}(t) \| \quad \text{where} \quad K = \frac{b}{1-(a+b+c)} \leq \frac{1}{2} \]

In general
\[ \Rightarrow \| g_n(t) - g_{n+1}(t) \| \leq K \| g_{n-1}(t) - g_n(t) \| \]
\[ \Rightarrow \| g_n(t) - g_{n+1}(t) \| \leq K^n \| g_0(t) - g_1(t) \| \quad \text{for all} \quad t \in \Omega \]

(2)

Now, we shall prove that for \( t \in \Omega \), \( \{g_n(t)\} \) is a Cauchy sequence. For this for every positive integer \( p \) we have, for \( t \in \Omega \)
\[ \Rightarrow \| g_n(t) - g_{n+p}(t) \| = \| g_n(t) - g_{n+1}(t) + g_{n+1}(t) - \ldots \ldots + g_{n+p-1}(t) - g_{n+p}(t) \| \]
\[ \leq \| g_n(t) - g_{n+1}(t) \| + \ldots + \| g_{n+p-1}(t) - g_{n+p}(t) \| \]
\[ \leq K^n [1 + K + K^2 + \ldots + K^{p-1}] \| g_0(t) - g_1(t) \| \]
\[ \leq \frac{K^n}{1-K} \| g_0(t) - g_1(t) \| \quad \text{for all} \quad t \in \Omega \]

as \( n \to \infty \), \( \| g_n(t) - g_{n+p}(t) \| \to 0 \), it follows that for \( t \in \Omega \), \( \{g_n(t)\} \) is a Cauchy sequence and hence is convergent in Hilbert space \( H \).

For \( t \in \Omega \), let
\[ \{g_n(t)\} \to g(t) \quad \text{as} \quad n \to \infty \]

(3)

Since \( C \) is closed, \( g \) is a function from \( C \) to \( C \).

Existence of random fixed point: For \( t \in \Omega \),

\[ \| g(t) - T(t, g(t)) \|^2 = \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \leq 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 + 2 \| \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]
\[ = 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 + 2 \| \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]
\[ \leq 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 + 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]
\[ \leq 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 + 2 \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]
\[ + 2b \| \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]
\[ + 2b \| g(t) - \sum_{n=0}^{\infty} T(t, g(t)) \|^2 \]

Therefore, \( g(t) = \sum_{n=0}^{\infty} T(t, g(t)) \) is a fixed point of \( T(t, g(t)) \) for \( t \in \Omega \).
\[+2c[\|S(t, g_{2n}(t)) - T(t, g(t))\|^2] = 2\|g(t) - g_{2n+1}(t)\|^2 + \frac{2c\|g(t) - T(t, g(t))\|^2}{1 + \|g_{2n}(t) - g(t)\|^2} + 2b[\|g_{2n}(t) - g_{2n+1}(t)\|^2 + \|g(t) - T(t, g(t))\|^2] + 2c[\|g_{2n+1}(t) - T(t, g(t))\|^2] \]

As \(\{g_{2n+1}(t)\}\) and \(\{g_{2n+2}(t)\}\) are subsequences of \(\{g_n(t)\}\) as \(n \to \infty\), \(g_{2n+1}(t) \to g(t)\) and \(g_{2n+2}(t) \to g(t)\).

Therefore,
\[
\Rightarrow \|g(t) - T(t, g(t))\|^2 \leq 2\|g(t) - g(t)\|^2 + \frac{2c\|g(t) - T(t, g(t))\|^2}{1 + \|g(t) - g(t)\|^2} + 2b[\|g(t) - g(t)\|^2 + \|g(t) - T(t, g(t))\|^2] + 2c[\|g(t) - T(t, g(t))\|^2] \]
\[
\Rightarrow [1 - 2(a + b + c)]\|g(t) - T(t, g(t))\|^2 \leq 0 \]
\[
\Rightarrow \|g(t) - T(t, g(t))\|^2 = 0 \quad (\text{as } 2(a+b+c)<1)
\]
\[
\Rightarrow T(t, g(t)) = g(t) \quad \forall t \in \Omega
\]

(4)

In an exactly similar way we can prove that for all \(t \in \Omega\),
\[
S(t, g(t)) = g(t)
\]

(5)

Again if \(A: \Omega \times \mathcal{C} \to \mathcal{C}\) is a continuous random operator on a non empty subset \(\mathcal{C}\) of a separable Hilbert space \(H\), then for any measurable function \(f: \Omega \to \mathcal{C}\), the function \(h(t) = A(t, f(t))\) is also measurable [1].

It follows from the construction of \(\{g_n(t)\}\) (by (1)) and the above consideration that \(\{g_n(t)\}\) is a sequence of measurable function. From (3), it follow that \(g\) is also a measurable function. This fact along with (4) and (5) shows that \(g: \Omega \to \mathcal{C}\) is common random fixed point of \(S\) and \(T\).
Uniqueness:

Let $h: \Omega \rightarrow C$ be another random fixed point common to $S$ and $T$ that is for $t \in \Omega$.

$S(t, h(t)) = h(t)$

$T(t, h(t)) = h(t)$

Then for $t \in \Omega$,

$$
\|g(t) - h(t)\|^2 \leq \|S(t, g(t)) - T(t, h(t))\|^2 + b[\|g(t) - S(t, g(t))\|^2 + \|h(t) - T(t, h(t))\|^2] + c[\|S(t, g(t)) - T(t, h(t))\|^2] \\
+ \frac{c([\|S(t, g(t)) - T(t, h(t))\|^2)}{1 + \|g(t) - b(t)\|^2}
$$

This complete proof of theorem.

References


Received: August, 2011