On Two-Parameter N-Times Integrated Semigroups

M. Khanehgir\textsuperscript{1}, F. Roohei and E. Goharei Moghaddam

Department of Mathematics, Faculty of Sciences
Islamic Azad University-Mashhad Branch
P.O. Box 413-91735, Mashhad-Iran

Abstract

This paper is concerned with the two-parameter n-times integrated semigroups on a sequentially complete locally convex space. Generator of a non-degenerate two-parameter n-times integrated semigroup is characterized and as an application we discuss the existence of solution of the abstract Cauchy problem

\[
\begin{align*}
\frac{\partial}{\partial t_1} U(t_1, t_2) &= H_1 U(t_1, t_2), & t_i &\in [0, a_i], \\
U(0, 0) &= x, & x &\in X, \ i = 1, 2.
\end{align*}
\]

Where \((H_1, H_2)\) is the infinitesimal generator of a two-parameter n-times integrated semigroup.

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1 Introduction

A generalization of \(C_0\)–semigroups is the so-called n-times integrated semigroups. A 0-times integrated semigroup is just a \(C_0\)–semigroup. This generalization semigroup was introduced by Arendt [1], and further developed by Kellermann and Hieber [4], Neubrander ([6],[7]). Arendt applied n-times integrated semigroup for solving the abstract Cauchy problem associated to non-densely defined linear operators. This theory is established in the setting of a Banach space. In [4] the concept of one-parameter integrated semigroups have been generalized to two-parameter integrated semigroups and the generator of such semigroups has been characterized. Also by using this semigroups,\textsuperscript{1}

\textsuperscript{1}khanehgir@mshdiau.ac.ir, mkhaneghir@gmail.com
the one-parameter abstract Cauchy problem associated to non-densely defined linear operators has been extended to the two-parameter case. This paper aims to study a natural generalization of two-parameter integrated semigroups to the two-parameter n-times integrated semigroups and the characterization of their generators. Finally we discuss the existence of solution of the abstract Cauchy problem. The ground space will be a sequentially complete locally convex space.

2 Preliminary Notes

Let $X$ be a sequentially complete locally convex space and let $L(X)$ denotes the space of all continuous linear operators on $X$.

Definition 2.1 Let $n \in \mathbb{N}$. A strongly continuous family $\{S(t)\}_{t \geq 0} \subset L(X)$ is called n-times integrated semigroup on $X$ if it satisfies:

(i) $S(0) = 0$,
(ii) $S(t)S(s) = \frac{1}{(n-1)!}\int_t^{s+t}(s+t-r)^{n-1}S(r)dr - \int_0^s(s+t-r)^{n-1}S(r)dr$.

An n-times integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

Finally $\{S(t)\}_{t \geq 0}$ is called exponentially equicontinuous if there is $\omega \in \mathbb{R}$ such that $\{e^{-\omega t}S(t) : t \geq 0\}$ is equicontinuous.

Suppose that $\{S(t)\}_{t \geq 0}$ is an n-times integrated semigroup on sequentially complete locally convex space $X$ such that $\{e^{-\omega t}S(t) : t \geq 0\}$ is equicontinuous for some $\omega \in \mathbb{R}$ (exponentially equicontinuous). Let $R_n : (\omega, \infty) \to L(X)$ be defined by

$$R_n(\lambda) x = \int_0^\infty \lambda^n e^{-\lambda t} S(t) x \, dt \quad (x \in X, \lambda > \omega).$$

Then $\ker R_n(\lambda)$ is the independent of $\lambda > \omega$. Hence by uniqueness theorem $R_n(\lambda)$ is injective if and only if $\{S(t)\}_{t \geq 0}$ is non-degenerate. In this case there is a unique operator $A$ satisfying $(\omega, \infty) \subset \rho(A)$ such that $R_n(\lambda) = (\lambda - A)^{-1}$ for all $\lambda > \omega$. This operator is called the generator of $\{S(t)\}_{t \geq 0}$. In fact the operator $A$ is defined by

$$D(A) = \{x \in X : x \in \text{Range}(R_n(\lambda))\},$$

$$A(x) = [\lambda - R_n(\lambda)]^{-1}(x) \quad (x \in D(A)).$$

On the other hand the linear operator $A$ is the generator of an n-times integrated semigroup if and only if $(\omega, \infty) \subset \rho(A)$ for some $\omega \in \mathbb{R}$ and the function $\lambda \to \frac{(\lambda - A)^{-1}}{\lambda^n}$ is a Laplace transform. By Laplace transform, it is routine to check that $A$ is the infinitesimal generator of only an n-times integrated semigroup $\{S(t)\}_{t \geq 0}$. 
Now we recall the concept of two-parameter integrated semigroup which is a generalization of the one-parameter integrated semigroup. Let \( \{W(t,s)\}_{t,s \geq 0} \) be a \( C_0 \)-two-parameter semigroup on a Banach space \( X \). We define a family \( \{S(t_1,s_1)\}_{t_1,s_1 \geq 0} \) of bounded linear operators on \( X \) as follows:

\[
S(t_1,s_1) = \int_0^{t_1} W(t,0)dt + \int_0^{s_1} W(0,s)ds.
\]

The family \( \{S(t_1,s_1)\}_{t_1,s_1 \geq 0} \) have the following properties:

(i) \( S(0,0) = 0 \),

(ii) \( (t_1,s_1) \rightarrow S(t_1,s_1) \) is strongly continuous,

(iii) \( S(t_1,s_1)S(t_2,s_2) = \int_0^{t_1} S(t + t_2,0) - S(t,0)dt + S(t_1,0)S(0,s_2) + S(t_2,0)S(0,s_1) + \int_0^{s_1} S(0,s_2 + s) - S(0,s)ds + [S(t_1,0) + S(0,s_1)]S(t_2,0) + S(0,s_2) \]

(iv) \( S(t_1,0)S(0,s_1) = S(0,s_1)S(t_1,0) \).

We say that a family \( \{S(t_1,s_1)\}_{t_1,s_1 \geq 0} \) of bounded linear operators on a Banach space \( X \) with the properties (i) – (iv) is a two-parameter integrated semigroup. In this case \( \{S(t_1,0)\}_{t_1 \geq 0} \) and \( \{S(0,s_1)\}_{s_1 \geq 0} \) are one-parameter integrated semigroups. For more details on this issue one can see [4]. In the next section we generalize this notion to two-parameter \( n \)-times integrated semigroup and Furthermore some of the properties of such semigroups are studied.

3 Main Results

As previously we mentioned let \( X \) be a sequentially complete locally convex space and \( L(X) \) be the space of all continuous linear operators on \( X \).

**Definition 3.1** Let \( n \in \mathbb{N} \). A family \( \{S(t,s)\}_{t,s \geq 0} \subset L(X) \) is called a two-parameter \( n \)-times integrated semigroup on \( X \) if it satisfies:

(i) \( S(0,0) = 0 \),

(ii) \( (t,s) \rightarrow S(t,s) \) is strongly continuous,

(iii) \( S(t_1,s_1)S(t_2,s_2) = \frac{1}{(n-1)!} \int_0^{t_1+t_2} \int_0^{s_1+s_2} - \int_0^{t_2}(t_1 + t_2 - r)^{n-1} S(r,0)dr + S(t_1,0)S(0,s_2) \]

\[
+ S(t_2,0)S(0,s_1) + \frac{1}{(n-1)!} \int_0^{s_1+s_2} - \int_0^{s_2}(s_1 + s_2 - r)^{n-1} S(0,r)dr,
\]

(iv) \( S(t,0)S(0,s) = S(0,s)S(t,0) \).

Trivially if \( \{S(t,s)\}_{t,s \geq 0} \) is a two-parameter \( n \)-times integrated semigroup then \( \{S(t,0)\}_{t \geq 0} \) and \( \{S(0,s)\}_{s \geq 0} \) are one-parameter \( n \)-times integrated semigroups.

By condition (iii) and (iv) it is easily can be checked that
\[ S(t_1, s_1) \cdot S(t_2, s_2) = S(t_1, 0)S(t_2, 0) + S(t_1, 0) \cdot S(0, s_2) \\
+ S(t_2, 0)S(0, s_1) + S(0, s_1) \cdot S(0, s_2) \\
= [S(t_1, 0) + S(0, s_1)][S(t_2, 0) + S(0, s_2)]. \]

and by induction for \( n = 2, 3, 4, \ldots \) we have
\[ S(t, s)^n = (S(t, 0) + S(0, s))^n. \]

A two-parameter \( n \)-times integrated semigroup \( \{S(t, s)\}_{t, s \geq 0} \) is said to be non-degenerate if \( S(t, s) \) \( x = 0 \) for all \( t, s \geq 0 \) implies \( x = 0 \).

Finally, \( \{S(t, s)\}_{t, s \geq 0} \) is called exponentially equicontinuous if there are \( \omega_1, \omega_2 \in \mathbb{R} \) such that \( \{e^{-\omega_1 t} S(t, 0)\} \) and \( \{e^{-\omega_2 t} S(0, s)\} \) are equicontinuous.

**Example 3.2** Let \( \{T_1(t) : t \geq 0\} \) and \( \{T_2(s) : s \geq 0\} \) are two one-parameter \( n \)-times integrated semigroups. It is easily can be checked that
\[ S(t, s) = T_1(t) + T_2(s) \]
is a two-parameter \( n \)-times integrated semigroup if and only if
\[ T_1(t) T_2(s) = T_2(s) T_1(t). \]

**Example 3.3** Let \( \{W(t, s)\}_{t, s \geq 0} \) be a \( C_0 \)-two-parameter semigroup with infinitesimal generator \( (H_1, H_2) \). There exist \( M, \omega_1, \omega_2 \in \mathbb{R} \) such that
\[ \|W(t, s)\| \leq M e^{\omega_1 t + \omega_2 s} \quad (t, s \geq 0). \]

For \( n \in \mathbb{N} \), let
\[ \int_0^t \frac{(t-r)^{n-1}}{(n-1)!} W(r, 0) \, dr \quad (t \geq 0) \]
and
\[ \int_0^s \frac{(s-r)^{n-1}}{(n-1)!} W(0, r) \, dr \quad (s \geq 0) \]
then by integrating \( n \)-times from \( R(\lambda, H_1) = \int_0^\infty e^{-\lambda r} W(r, 0) dr \) it yields that:
\[ \frac{R(\lambda, H_1)}{\lambda^n} = \int_0^\infty e^{-\lambda t} S^n(t, 0) \, dt \quad (\lambda > \max \{0, w_1\}) \]
and similarly
\[ \frac{R(\lambda, H_2)}{\lambda^n} = \int_0^\infty e^{-\lambda s} S^n(0, s) \, ds \quad (\lambda > \max \{0, w_2\}) . \]

Now \( S^n(t, s) = S^n(t, 0) + S^n(0, s) \) is a two-parameters \( n \)-times integrated semigroup.

**Example 3.4** Let \( \{S(t, s)\}_{t, s \geq 0} \) be a two-parameter \( (n - 1) \)-times integrated semigroup. Then
\[ \tilde{S}(t, s) = \int_{0}^{t_1} S(t, 0) \, dt + \int_{0}^{s_1} S(0, s) \, ds \]
is a two-parameter \(n\)-times integrated semigroup. Since
\[
\tilde{S}(t_1, s_1) \tilde{S}(t_2, s_2) = \left[ \int_0^{t_1} S(t, 0) \tilde{S}(t_2, s_2) x \, dt + \int_0^{s_1} S(0, s) \tilde{S}(t_2, s_2) x \, ds \right]
= \left[ \int_0^{t_1} S(t, 0) \left( \int_0^{t_2} S(t', 0) x \, dt' + \int_0^{s_2} S(0, s') x \, ds' \right) dt \right]
+ \left[ \int_0^{s_1} S(0, s) \left( \int_0^{t_2} S(t', 0) x \, dt' + \int_0^{s_2} S(0, s') x \, ds' \right) ds \right]
= \tilde{S}(t_1, 0) \tilde{S}(t_2, 0) x + \tilde{S}(t_1, 0) \tilde{S}(0, s_2) x
+ \tilde{S}(0, s_1) \tilde{S}(t_2, 0) x + \tilde{S}(0, s_1) \tilde{S}(0, s_2) x
= \frac{1}{(n-1)!} \int_0^{t_1+t_2} - \int_0^{t_1} - \int_0^{t_2} (t_1 + t_2 - r)^{n-1} \tilde{S}(r, 0) x \, dr
+ \tilde{S}(t_1, 0) \tilde{S}(0, s_2) + \tilde{S}(0, s_1) \tilde{S}(t_2, 0)
+ \frac{1}{(n-1)!} \int_0^{s_1+s_2} - \int_0^{s_1} - \int_0^{s_2} (s_1 + s_2 - r)^{n-1} \tilde{S}(0, r) x \, dr.
\]

Now suppose \(\{S(t, s)\}_{t,s \geq 0}\) is a non-degenerate exponentially equicontinuous two-parameter \(n\)-times integrated semigroup and let \(H\) and \(K\) be the generator of \(n\)-times integrated semigroups \(\{S(t, 0)\}_{t \geq 0}\) and \(\{S(0, s)\}_{s \geq 0}\) respectively. We shall think of \((H, K)\) as the generator of \(\{S(t, s)\}_{t,s \geq 0}\). By using Fubini’s theorem one can prove that \(R^1_n(\lambda)\) commutes with \(R^2_n(\mu)\) where
\[
R^1_n(\lambda) = \int_0^\infty \lambda^n e^{-\lambda t} S(t, 0) \, dt \quad (\lambda \in D(R^1_n))
\]
and
\[
R^2_n(\mu) = \int_0^\infty \mu^n e^{-\mu s} S(0, s) \, ds \quad (\mu \in D(R^2_n)).
\]

**Theorem 3.5** If \((H_1, H_2)\) is the generator of a non-degenerate exponentially equicontinuous two-parameter \(n\)-times integrated semigroup \(\{S(t, s)\}_{t,s \geq 0}\) then for all \(x \in D(H_1) \cap D(H_2)\), \(S(t, s) x \in D(H_1) \cap D(H_2)\) and moreover
\[
H_i S(t, s) x = S(t, s) H_i x, \quad i = 1, 2.
\]

**Proof.** We prove this for \(i = 1\). It is enough to show that
\[
R^1_n(\lambda) S(t, s)x = S(t, s) R^1_n(\lambda)x.
\]
This is true since
\[
R^1_n(\lambda) S(t, s)x = \int_0^\infty e^{-\lambda r} S(r, 0) S(t, s)x \, dr = S(t, s) \int_0^\infty e^{-\lambda r} S(r, 0)x \, dr = S(t, s) R^1_n(\lambda).
\]
Theorem 3.6 If \((H_1, H_2)\) is the generator of an exponentially equicontinuous non-degenerate two-parameter \(n\)-times integrated semigroup \(\{S(t,s)\}_{t,s \geq 0}\) then the following conditions are equivalent:

(i) \(S(t,s) x = S(t,0) x + S(0,s) x\) for all \(x \in X\),

(ii) If \((H_1, H_2)\) is the generator of another exponentially equicontinuous two-parameter \(n\)-times integrated semigroup \(\{S'(t,s)\}_{t,s \geq 0}\) then \(S(t,s) = S'(t,s)\) for all \(t, s \geq 0\). Moreover, if for some \(t, s \geq 0\); \(S(t,s)\) is injective then condition (i) can be obtain easily.

Proof. It is proved exactly like two-parameter integrated semigroups (See [4]).

In the following theorem we have a characterization of generator with dense domain of a two-parameter \(n\)-times integrated semigroup.

Theorem 3.7 Assume that \(n \in \mathbb{N}\). The family \(\{S(t,s)\}_{t,s \geq 0}\) is a two-parameter \((n+1)\)-times integrated semigroup with generator \((H,K)\), with dense domain in \(X\) satisfying:

\[
\lim \sup_{h \to 0} \frac{1}{h} \| S(t+h,0) - S(t,0) \| \leq M_1 e^{\omega_1 t} \quad (t \geq 0)
\]

and

\[
\lim \sup_{k \to 0} \frac{1}{k} \| S(0,s+k) - S(0,s) \| \leq M_2 e^{\omega_2 s} \quad (s \geq 0) \quad (*)
\]

if and only if there exist \(a_1 \geq \max \{\omega_1, 0\}\) and \(a_2 \geq \max \{0, \omega_2\}\) such that \((a_1, \infty) \subset \rho(H)\) and \((a_2, \infty) \subset \rho(K)\) and

\[
(i) \| \frac{(\lambda - \omega_1)^{k+1}}{k!} \left[ \frac{R(\lambda, H)}{\lambda^n} \right]^{(k)} \| \leq M_1 \quad \text{for all} \ \lambda > a_1, \ k = 0, 1, 2, \ldots
\]

\[
(ii) \| \frac{(\mu - \omega_2)^{k+1}}{k!} \left[ \frac{R(\mu, K)}{\mu^n} \right]^{(k)} \| \leq M_2 \quad \text{for all} \ \mu > a_2, \ k = 0, 1, 2, \ldots
\]

\[(**)
\]

(iii) \(R(\lambda, H)R(\mu, K) = R(\mu, K)R(\lambda, H)\) for \(\lambda \in \rho(H)\) and \(\mu \in \rho(K)\).

Proof. (i) and (ii) obtain from Theorem (4.1) of [1] and (iii) obtains by Fubini’s theorem as we mentioned before. Conversely by the same theorem \(H\) is the generator of the \(n\)-times integrated semigroup \(\{T_1(t)\}_{t \geq 0}\) and \(K\) is the generator of the \(n\)-times integrated semigroup \(\{T_2(s)\}_{s \geq 0}\). Now by example 3.2 and since \(H\) and \(K\) are densely defined so \(S(t,s) = T_1(t) + T_2(s)\) is a two-parameter \(n\)-times integrated semigroup where \(T_1(t) = S(t,0)\) and \(T_2(s) = S(0,s)\) and conditions (*) hold.
Now let the equivalent condition (*) and (**) be satisfied. In general if the assumption densely defined of the generators removed then let $H_1$ be the part of $H$ in $\overline{D(H)}$ and $K_1$ be the part of $K$ in $\overline{D(K)}$. As we mentioned by Theorem (4.1) of [1], $H_1$ is the generator of an $n$-times integrated semigroup $\{T_1(t)\}_{t \geq 0}$ and $H_2$ is the generator of an $n$-times integrated semigroup $\{T_2(s)\}_{s \geq 0}$. Hence $(H_1, K_1)$ is the generator of a two-parameter $n$-times integrated semigroup

$$W(t, s) = T_1(t) + T_2(s)$$
on $\overline{D(H_1)} \cap \overline{D(K_1)}$.

**Corollary 3.8** If $\{S(t, s)\}_{t, s \geq 0}$ is a two-parameter integrated semigroup then $(H_1, K_1)$ defined as above is the generator of a strongly continuous two-parameter semigroup

$$W(t, s) = T_1(t) T_2(s)$$
on $\overline{D(H_1)} \cap \overline{D(K_1)}$.

**Corollary 3.9** If $(H_1, H_2)$ is the generator of a two-parameter $(n+1)$-times integrated semigroup $\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}$ then $S(t_1, t_2)x = S(t_1, 0)x + S(0, t_2)x$ for all $x \in \bigcap_{i=1}^{2} \overline{D(H_i)}$. For this, suppose $H_i$ is the part of $H_1$ in $\overline{D(H_1)}$ for $i = 1, 2$. By the assertion before corollary (3.8) $(H_1, H_2)$ is the generator of a two-parameter $n$-times integrated semigroup $\{W(t_1, t_2)\}_{t_1, t_2 \geq 0}$ in which

$$S(t_1, t_2)x = \int_{0}^{t_1} W(t, 0)x + \int_{0}^{t_2} W(0, s)x$$

for all $x \in \bigcap_{i=1}^{2} \overline{D(H_i)}$. But the right-hand side is equal to $S(t_1, 0)x + S(0, t_2)x$.

We terminate this paper with the abstract Cauchy problem. Consider the abstract Cauchy problem (2-Acp)

$$\begin{align*}
\frac{\partial}{\partial t_i} U(t_1, t_2) &= H_i U(t_1, t_2), \quad t_i \in [0, a_i]; \\
U(0, 0) &= x, \quad x \in X, \; i = 1, 2,
\end{align*}$$

Where $(H_1, H_2)$ is assumed to be the generator of a non-degenerate $n$-times two-parameter integrated semigroup $\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}$. A solution of 2-Acp is a function

$$U(t_1, t_2) \in C^1([0, a_1] \times [0, a_2], \bigcap_{i=1}^{2} \overline{D(H_i)})$$

$$= \{u : u \text{ is continuously differentiable and } U([0, a_1] \times [0, a_2]) \subseteq \bigcap_{i=1}^{2} \overline{D(H_i)}\}$$

such that 2 - Acp holds.

Let $V : [0, a_1] \times [0, a_2] \to X$ be the function given by

$$V(t_1, t_2) = S(t_1, t_2)x \quad 0 \leq t_1 \leq a_1, \; 0 \leq t_2 \leq a_2.$$
Theorem 3.10 For given \( x \in X \), \( 2 - Acp \) has a solution if and only if
\[
\frac{\partial^n}{\partial t^n_{2}} S(0, t_2) \ x \in C^1([0, a_2], X)
\]
and
\[
\frac{\partial^n}{\partial t^n_{1}} S(t_1, 0) \left( \frac{\partial^n}{\partial t^n_{2}} S(0, t_2) \ x \right) \in C^1([0, a_1], X).
\]
In this case, the function
\[
U(t_1, t_2) = \frac{\partial^n}{\partial t^n_{1}} S(t_1, 0) \circ \frac{\partial^n}{\partial t^n_{2}} S(0, t_2) \ x
\]
is the unique solution of \( 2 - Acp \).

**Proof.** We know that for \( t_1, t_2 \geq 0 \),
\[
S(t_1, 0) \ S(0, t_2) = S(0, t_2) \ S(t_1, 0).
\]
By taking derivative with respect to \( t_1 \) and by continuity of \( S(0, t_2) \) we have
\[
\frac{\partial}{\partial t_1} S(t_1, 0) \ S(0, t_2) \ x = S(0, t_2) \ \frac{\partial}{\partial t_1} S(t_1, 0) \ x.
\]
Now by taking derivative with respect to \( t_2 \) of above assertion we conclude that
\[
\frac{\partial}{\partial t_2} S(0, t_2) \ \frac{\partial}{\partial t_1} S(t_1, 0) \ x = \frac{\partial}{\partial t_1} S(t_1, 0) \ \frac{\partial}{\partial t_2} S(t_2, 0) \ x.
\]
By repeating this process we obtain:
\[
\frac{\partial^n}{\partial t^n_{1}} S(t_1, 0) \circ \frac{\partial^n}{\partial t^n_{2}} S(0, t_2) = \frac{\partial^n}{\partial t^n_{2}} S(0, t_2) \circ \frac{\partial^n}{\partial t^n_{1}} S(t_1, 0).
\]
Now by this, it easily to check that
\[
U(t_1, t_2) = \frac{\partial^n}{\partial t^n_{1}} S(t_1, 0) \circ \frac{\partial^n}{\partial t^n_{2}} S(0, t_2) \ x
\]
is a solution of \( 2 - Acp \).

**References**


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