Study of a Prey-Predator Model with Diseased Prey

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Abstract

A mathematical model is proposed and analyzed to study the diseased prey in a Lotka Volterra prey-predator model. It has been assumed that the susceptible and infected prey populations are predated by predator species. The criteria for the feasibility of equilibrium for the model are obtained. A remarkable observation about one of the equilibrium point $E_3$ is that in the absence of diseased prey, the $x_1$ coordinate goes at higher level in comparison with that of sinha et. al. [13]. This increase is about-100% when the new parameter $e$ is 0.5. Such observation can also be seen for non zero equilibrium point $E_4$. Defining Lyapunov function global stability is also established.
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1. Introduction

   The Lotka Volterra model is one of the earliest prey-predator models which is based on sound mathematical logic [6]. It forms the basis of many models used today in population dynamics [2, 3, 4, 9, 11, 15, 16]. The interaction of one prey and one predator is well understood. Mathematically the investigation of such models is simplified by the fact that they are two dimensional and thus Poincare- Bendixon theorem applies.

   As the dimension increases the analysis becomes complex in nature. Models for the interaction of more than two species have been studied in recent years. Many investigators have concentrated on the persistence problems. In biological terms persistence states that asymptotically the density of each species remains above a positive bond independent of initial conditions, which means that the species stay away from extinction. Mathematically this may be stated in terms of behavior of solutions of the models representing biological phenomenon [7]. The effect of self regulation in a host pathogen model and microbial pest control is analyzed by Begon and Bowers [10]. A prey-predator model with diffusion and a supplementry resource for the prey in a two patch environment is studied by Dhar [8]. Stability analysis of a prey-predator model with time delay and harvesting are the subject of analysis of Taha [14] and Martin [1].

   In a recent communication Sinha et. al. analyzed two models to study the prey-predator dynamics under the simultaneous effect of toxicant and disease. In model I they have proposed a prey-predator model in which prey population is affected by disease. It has been assumed that the susceptible and infected prey populations are being predated by predator.
Prey-predator model with diseased prey

In the present paper we have reframed and analyzed this model for a more realistic situation. The mathematical formulation is more general and thus includes the model I of Sinha et. al [13] as a corollary.

2. Mathematical model and Equilibrium points

Let us assume that \( x_1 \) (t) and \( x_2 \) (t) are susceptible and infected parts of a prey population \( y(t) \) represents the predator population which is fed by both susceptible and infected prey population with different predation rates. Assuming \( e \) as the coefficient of the biomass adopted a by predator, the mathematical model proposed for present study is in the following form

\[
\begin{align*}
\frac{dx_1}{dt} &= \theta - \beta x_1 x_2 - \alpha_1 x_1 y - d_1 x_1 \\
\frac{dx_2}{dt} &= \beta x_1 x_2 - \alpha_2 x_2 y - d_2 x_2 \\
\frac{dy}{dt} &= e \alpha_1 x_1 y + e \alpha_2 x_2 y - d_3 y \\
\end{align*}
\]

\( \text{.................. (2.1)} \)

where

\( \beta \) : disease contact rate

\( \theta \) : constant recruitment rate of susceptible prey

\( \alpha_1, \alpha_2 \) : predation rates among the susceptible and infected prey populations

\( d_1, d_2, d_3 \) : natural death rates of respective population

We shall use the initial conditions \( x_1(0) = x_{10} > 0, x_2(0) = x_{20} > 0 \) and \( y(0) = y_0 > 0 \)
The developed model follows other assumptions of model I given by Sudipa Sinha et. al. [13].

In order to prove the boundedness of solutions of (2.1) we need the following Lemma:

Lemma: All solutions of model given by (2.1) will lie in the region

\[ G = \left\{ (x_1, x_2, y) \in \mathbb{R}^3 : 0 \leq x_1 + x_2 + y \leq \frac{\theta}{\theta_1} \right\} \text{ as } t \to \infty \] for all positive initial values \((x_{10}, x_{20}, y_0) \in \mathbb{R}^3\) where \(\theta_1 = \min (d_1, d_2, d_3)\)

Proof: Let us consider the following function

\[ w(t) = x_1(t) + x_2(t) + y(t) \]

so that

\[ w(t) = x_1(t) + x_2(t) + y(t) = \theta - \theta_1 w \] where \(\theta_1 = \min (d_1, d_2, d_3)\) then by usual comparison theorem [16] we get the following expression as \(t \to \infty\)

\[ w(t) \leq \frac{\theta}{\theta_1} \]

Thus \(x_1(t) + x_2(t) + y(t) \leq \frac{\theta}{\theta_1}\), which proves the Lemma.

Equating the right hand side of (2.1) to zero the following equilibrium points can be obtained.

\[ E_0(0, 0, 0), \ E_1 \left( \frac{\theta}{d_1}, 0, 0 \right), \ E_2 \left( \frac{d_2}{\beta}, \frac{\beta \theta - d_1 d_2}{\beta d_2}, 0 \right), \ E_3 \left( \frac{d_1}{\alpha_1}, 0, \frac{\theta d_1 - d_2}{\alpha_1 d_1} \right) \]

and \[ E_4 \left( x_1^*, x_2^*, y^* \right) \]

where \(x_1^* = \frac{d_1 - e \alpha_1 x_1^*}{e \alpha_1}\),

\[ x_2^* = \frac{d_2}{e \alpha_2} + \frac{e \theta \alpha_1}{e \alpha_1 d_2 - e \alpha_2 d_1 - \beta d_3} \]

and
Prey-predator model with diseased prey

3. Equilibrium Analysis

The variation matrix for the system (2.1) is

\[
V = \begin{bmatrix}
-\beta x_2 - \alpha y - d_1 - \lambda & -\beta x_1 & -\alpha x_1 \\
\beta x_2 & \beta x_1 - \alpha_2 y - d_2 - \lambda & -\alpha_2 x_2 \\
e^{x_1}y & e^{\alpha_2}y & e^{\alpha_1}x_1 + e^{\alpha_2}x_2 - d_3 - \lambda
\end{bmatrix}
\]  

The dynamical behaviour of model (2.1) is as follows:

(a) \( E_0 (0, 0, 0) \) is a trivial equilibrium point. All the eigen values are negative showing that it is locally stable.

(b) The eigen values corresponding to the equilibrium point \( E_1 (\theta/d_1, 0, 0) \) are \( \beta \theta - d_1 d_2 / d_1 \) and \( e^{\alpha_1} \theta - d_1 d_3 / d_1 \). Thus it will be locally stable if \( \beta \theta < d_1 d_2 \) and \( e^{\alpha_1} \theta < d_1 d_3 \). Further this is always stable in \( x_1 \) direction. It shows an unstable saddle point for \( \beta \theta > d_1 d_2 \) in \( x_1 x_2 \) direction and \( e^{\alpha_1} \theta > d_1 d_3 \) in \( x_1 y \) plane.

(c) The equilibrium point \( E_2 \) will biologically exist if \( \beta \theta > d_1 d_2 \) and

\[
\theta > d_1 d_2 / e^{\alpha_1} \quad (3.3)
\]

one of the eigen value is \( e^{\alpha_2} d_2 / \beta \) + \( e^{\alpha_2} \left( \beta \theta - d_1 d_2 / \beta d_2 \right) - d_3 \) \( (3.3) \)

and other two are given by the roots of the quadratic equation.
\[ \lambda^2 + \left[ \frac{\beta \theta - d_1 d_3}{d_2} + d_1 \right] \lambda + \left( \beta \theta - d_1 d_3 \right) = 0 \]  \hspace{1cm} (3.4) 

Thus it is attractive in \( x_1 \) direction if 

\[ d_3 < \left[ \frac{e \alpha_1 d_3}{\beta} + e \alpha_2 \beta \theta - d_1 d_3 \right] \]

and in this situation it shows a local stability.

(d) The equilibrium point 

\[ E_3 \left( \frac{d_3}{e \alpha_1}, 0, \frac{\theta e \alpha_1 - d_1 d_3}{e \alpha_1 d_3} \right) \]

needs a condition \( \theta e \alpha_1 > d_1 d_3 \) for its existence. The variation matrix (3.1) for this position provides the eigen values \( \lambda_1 = \beta x_1 - \alpha_2 y - d_2 \) and the two eigen values as the roots of the quadratic equation.

\[ \lambda^2 + (\alpha_1 y + d_1) \lambda + e \alpha_1^2 x_1 y = 0 \]

where

\[ \bar{x}_1 = d_1 / e \alpha_1 \] \hspace{1cm} and \hspace{1cm} \[ \bar{y} = \frac{\theta e \alpha_1 - d_1 d_3}{e \alpha_1 d_3} \]

(e) Non zero equilibrium \( E_4 \left( x_1^*, x_2^*, y^* \right) \) is feasible provided

\[ d_3 > e \alpha_2 x_2^* \]

\[ e \alpha_1 d_2 > e \alpha_2 d_1 + \beta d_3 \]

and \( \beta d_3 > e \alpha_2 d_2 + \beta e \alpha_2 x_2^* \) \hspace{1cm} (3.5) 

The eigen values corresponding to this point are given by the roots of the polynomial equation

\[ \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \]

where

\[ a_1 = \beta x_2^* + \alpha_2 y^* + d_1 \]

\[ a_2 = \beta^2 x_1^* x_2^* + e \alpha_1^2 x_1^* y^* + e \alpha_2^2 x_2^* y^* \]

\[ a_3 = e \alpha_2^2 x_2^* y^* \left( \beta x_2^* + \alpha_2 y^* + d_1 \right) \]
Prey-predator model with diseased prey

\[ x_1^* = \frac{d_1 - e\alpha_2 x_2^*}{e\alpha_1} > 0 \]
\[ x_2^* = \frac{d_3 + \frac{\theta e\alpha_1}{e\alpha_2 d_2 - e\alpha_2 d_1 - \beta d_3}}{e\alpha_1 d_2 - e\alpha_2 d_1 - \beta d_3} > 0 \]
\[ y^* = \frac{\beta d_3 - e\alpha_2 d_3 - \beta e\alpha_2 x_2^*}{e\alpha_1 d_2} > 0 \]

Here it is clear that under above conditions \( a_1 > 0, a_2 > 0 \) and \( a_3 > 0 \). Thus from Routh Hurwitz criteria it will be locally stable if \( a_1 a_3 > a_2 \).

\[ (4) \quad \text{Global stability of } E_4 \left( x_1^*, x_2^*, y^* \right) \]

In order to prove the global stability we define the Lyapunov function

\[ V(x_1, x_2, y) = \sum_{i=1}^{2} w_i \left( x_i - x_i^* - x_i^* \log \frac{x_i}{x_i^*} \right) + w_3 \left( y - y^* - y^* \log \frac{y}{y^*} \right) \]

The time derivative of above function will be \( \ldots \ldots \) (4.1)

\[ \dot{V}(t) = \omega_1 z_1 \frac{x_1}{x_1^*} + \omega_2 z_2 \frac{x_2}{x_2^*} + \omega_3 z_3 \frac{y}{y^*} \ldots \ldots \] (4.2)

where

\[ z_1 = (x_1 - x_1^*), \quad z_2 = (x_2 - x_2^*) \text{ and } z_3 = (y - y^*) \]

using the set of equations (2.1) and (4.2) we obtain

\[ \dot{V}(t) = -\theta \omega_1 z_1^2 \frac{1}{x_1 x_1^*} - \beta \omega_1 z_1 z_2 - \alpha_1 \omega_1 z_1 z_3 + \beta \omega_2 z_1 z_2 \]
\[ -\alpha_2 \omega_1 z_2 z_3 + e \alpha_1 \omega_1 z_1 z_3 + e \alpha_2 \omega_1 z_2 z_3 \]

Now choosing \( \omega_1 = \omega_2 = 1 \) and \( \omega_3 = \frac{1}{e} \) we get
\[ \dot{V}(t) = -\frac{\theta z^2_i}{x_i x_i^*} \leq 0 \]

This shows than \( E_4 \) is globally asymptotically stable. Hence the Lyapunov theorem implies that \( E_4 \left(x^*_1, x^*_2, y^*\right) \) is globally asymptotically stable.

5. Conclusion

The existence of trivial equilibrium is unconditional non existence of \( E_1 \) will imply that \( E_2 \) will also be biologically extinct. However the presence of \( E_3 \) will guarantee the unstability of \( E_1 \). Another important observation about \( E_3 \) is that it moves at higher level in \( x_1 \) direction as the new parameter get its value between \( 0 < e < 1 \). This increase is about 100% when \( e = 1/2 \). Such an observation can also be marked for \( E_4 \left(x^*_1, x^*_2, y^*\right) \).

For \( e = 1 \), the analysis coincides with Sinha et.al. [1].

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References


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