On Intuitionistic $Q$-Fuzzy $K$-Ideals of Semiring

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Abstract
In this paper, we apply the concept of intuitionistic $Q$-fuzzy set to semirings. We introduced the notion of anti $Q$-fuzzy right ideal, anti $Q$-fuzzy right $k$-ideal and intuitionistic $Q$-fuzzy right $k$-ideal in semiring. We investigate the their properties and connections with right $k$-ideal, $Q$-fuzzy right $k$-ideal, anti $Q$-fuzzy right $k$-ideal.

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1 Introduction

Henriksen defined in Henriksen (1958) a more restricted class of ideals in semiring, which is called the class of $k$-ideals, with the property that if the semiring $R$ is a ring then a complex in $R$ is a $k$-ideal if and only if it is a ring ideal.

Atanassov introduced intuitionistic fuzzy set which constitute a generalization of the notion of fuzzy sets [1],[2]. The degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. M. Akram and W.A. Dudek introduced the notion of intuitionistic fuzzy left $k$-ideal in semiring [4]. K.H. Kim[3] studied intuitionistic $Q$-fuzzy ideals. In this paper, we apply the concept of intuitionistic $Q$-fuzzy set to semirings. We introduced the notion of anti $Q$-fuzzy right ideal, anti $Q$-fuzzy right $k$-ideal and intuitionistic $Q$-fuzzy right $k$-ideal in semiring. We investigate the their properties and connections with right $k$-ideal, $Q$-fuzzy right $k$-ideal, anti $Q$-fuzzy right $k$-ideal.

2 Preliminary Notes

Definition 2.1 A nonempty set $R$ together with two binary operations $’+$’ and $’.’$ is said to be a semiring if
1) \((R; +)\) is a commutative semigroup,
2) \((R; \cdot)\) is a semigroup,
3) \(a(b + c) = ab + ac\) and \((a + b)c = ac + bc\) for all \(a, b, c \in R\).

By a subsemiring of \(R\) we mean a nonempty subset \(S\) of \(R\) such that \(S\) is closed under the operation of addition and multiplication in \(R\). A subsemiring \(R\) is called an right (left) ideal of \(R\) if for all \(r \in R, x \in I, xr \in I (rx \in I)\). A subsemiring \(I\) of a semiring \(S\) is called an ideal of \(R\) if it is both left and right ideal.

A mapping \(\mu : M \times Q \to [0, 1]\), where \(M, Q\) are arbitrary non-empty sets, is called a \(Q\)-fuzzy set of \(M\). An upper level set of a \(Q\)-fuzzy set \(\mu\) denoted by \(U(\mu; t) = \{x \in M \mid \mu(x, q) \geq t, \forall q \in Q\}\) and a lower level set of a \(Q\)-fuzzy set \(\mu\) denoted by \(L(\mu; t) = \{x \in M \mid \mu(x, q) \leq t, \forall q \in Q\}\), for all \(t \in [0, 1]\).

An intuitionistic \(Q\)-fuzzy set (IQFS for short) defined on non-empty sets \(M\) and \(Q\) as objects of the form

\[
A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > \mid x \in R, q \in Q\},
\]

where the function \(\mu_A : M \times Q \to [0, 1]\) and \(\lambda_A : M \times Q \to [0, 1]\) denote the degree of membership (namely \(\mu_A(x, q)\)) and the degree of non-membership (namely \(\lambda_A(x, q)\)) for each element \(x \in M, q \in Q\) to the set \(A\), respectively, and

\[
0 \leq \mu_A(x, q) + \lambda_A(x, q) \leq 1
\]

for each \(x \in M, q \in Q\). Obviously, every \(Q\)-fuzzy set \(\mu\) we can have

\[
A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > \mid x \in M, q \in Q\}.
\]

For the sake of simplicity, we shall use the symbol \(A = (\mu_A, \lambda_A)\) for the intuitionistic \(Q\)-fuzzy set \(A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > \mid x \in R, q \in Q\}\). Obviously for an IQFS \(A = (\mu_A, \lambda_A)\) in \(M\), when \(\lambda_A(x, q) = 1 - \mu_A(x, q)\), for every \(x \in M, q \in Q\), the IQFS \(A\) is a \(Q\)-fuzzy set.

### 3 Main Results

**Definition 3.1** A \(Q\)-fuzzy set \(\mu\) of a semiring \(R\) is said to be a \(Q\)-fuzzy right (left) ideal if

1) \(\mu(x + y, q) \geq \mu(x, q) \wedge \mu(y, q)\).
2) \(\mu(xy, q) \geq \mu(x, q)(\mu(xy, q) \geq \mu(y, q))\)

for all \(x, y \in R, q \in Q\).

**Definition 3.2** A \(Q\)-fuzzy set \(\mu\) of a semiring \(R\) is said to be an anti \(Q\)-fuzzy right (left) ideal if
1) \( \mu(x + y, q) \leq \mu(x, q) \wedge \mu(y, q) \).
2) \( \mu(xy, q) \leq \mu(x, q)(\mu(xy, q) \leq \mu(y, q)) \)
for all \( x, y \in R, q \in Q \).

**Definition 3.3** An intuitionistic \( Q \)-fuzzy set \( A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > | x \in R, q \in Q \} \) is called an intuitionistic \( Q \)-fuzzy right (left) ideal of \( R \) if
1) \( \mu_A(x + y, q) \geq \mu_A(x, q) \wedge \mu_A(y, q) \).
2) \( \mu_A(xy, q) \geq \mu_A(x, q)(\mu_A(xy, q) \geq \mu_A(y, q)) \).
3) \( \lambda_A(x + y, q) \leq \lambda_A(x, q) \vee \lambda_A(y, q) \).
4) \( \lambda_A(xy, q) \leq \lambda_A(x, q)(\lambda_A(xy, q) \leq \lambda_A(y, q)) \),
for all \( x, y \in R, q \in Q \).

**Theorem 3.4** If a \( Q \)-fuzzy set \( \mu \) is a \( Q \)-fuzzy right (left) ideal of a semiring \( R \) if and only if \( 1 - \mu \) is an anti-\( Q \)-fuzzy right (left) ideal in a semiring \( R \).

**Proof.** Let \( \mu \) be a \( Q \)-fuzzy right ideal in a semiring \( R \). Let \( x, y \in R, q \in Q \). Then \( \mu(x + y, q) \geq \mu(x, q) \wedge \mu(y, q) \) implies \(-\mu(x + y, q) \leq -(\mu(x, q) \wedge \mu(y, q)) \). Thus \(-\mu(x + y, q) \leq -(\mu(x, q) \vee -\mu(y, q)) \). Then \( 1 - \mu(x + y, q) \leq 1 - \mu(x, q) \vee 1 - \mu(y, q) \). Therefore \( (1 - \mu)(x + y, q) \leq (1 - \mu)(x, q) \vee (1 - \mu)(y, q) \).
Similarly, we can prove \( (1 - \mu)(xy, q) \leq \mu(x, q) \). Hence \( 1 - \mu \) is anti-\( Q \)-fuzzy right ideal in semiring \( R \). Conversely, we can prove that \( \mu \) is a \( Q \)-fuzzy right ideal in similar manner.

**Lemma 3.5** A \( Q \)-fuzzy set \( \mu \) in a semiring \( R \) is a \( Q \)-fuzzy right (left) ideal if and only if \( U(\mu, t) \) is a right (left) ideal of a semiring \( R \) for all \( t \in [0, 1] \) whenever nonempty.

**Definition 3.6** A right (left) ideal \( I \) is called right (left) \( k \)-ideal of a semiring \( R \) if \( x + y, y \in I \) implies \( x \in I \).

**Definition 3.7** A \( Q \)-fuzzy right (left) ideal \( \mu \) of a semiring \( R \) is called \( Q \)-fuzzy right (left) \( k \)-ideal if for all \( x, y \in R, q \in Q \),
\[
\mu(x, q) \geq \mu(x + y, q) \wedge \mu(y, q)
\]

**Definition 3.8** An anti-\( Q \)-fuzzy right (left) ideal \( \mu \) of a semiring \( R \) is called anti-\( Q \)-fuzzy right (left) \( k \)-ideal if for all \( x, y \in R, q \in Q \),
\[
\lambda(x, q) \leq \lambda(x + y, q) \vee \lambda(y, q)
\]

**Definition 3.9** An intuitionistic \( Q \)-fuzzy right (left) ideal \( A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > | x \in R, q \in Q \} \) in \( R \) is said to be an intuitionistic \( Q \)-fuzzy right (left) \( k \)-ideal if
1) \( \mu_A(x, q) \geq \mu_A(x + y, q) \wedge \mu_A(y, q) \)
2) \( \lambda_A(x, q) \leq \lambda_A(x + y, q) \vee \lambda_A(y, q) \)
for all \( x, y \in R, q \in Q \).
Theorem 3.10 If a $Q$-fuzzy set $\mu$ is a $Q$-fuzzy right $k$-ideal in a semiring $R$ if and only if $1 - \mu$ is an anti $Q$-fuzzy right (left) $k$-ideal in a semiring $R$.

Proof. Let $\mu$ be a $Q$-fuzzy right $k$-ideal in a semiring $R$. Let $x, y \in R, q \in Q$. Then $\mu(x, q) \geq \mu(x + y, q) \cap \mu(y, q)$ implies $-\mu(x, q) \leq -(\mu(x + y, q) \cap \mu(y, q))$. Thus $-\mu(x, q) \leq -\mu(x + y, q) \lor -\mu(y, q)$. Therefore $1 - \mu(x, q) \leq 1 - \mu(x + y, q) \lor 1 - \mu(y, q))$. Hence $(1 - \mu)(x, q) \leq (1 - \mu)(x + y, q) \lor (1 - \mu)(y, q)$. By Theorem 3.4, $1 - \mu$ is an anti $Q$-fuzzy right $k$-ideal. By similar argument, we can prove the converse part.

Lemma 3.11 A $Q$-fuzzy set $\mu$ of a semiring $R$ is a $Q$-fuzzy right $k$-ideal if and only if $U(\mu; t)$ is a right $k$-ideal in a semiring $R$ for all $t \in [0, 1]$ whenever nonempty.

Proof. Let $\mu$ be a $Q$-fuzzy right $k$-ideal of a semiring $R$. Let $t \in [0, 1]$. If there exists $x, y \in R, q \in Q$ such that $x + y, y \in U(\mu; t)$ and $x \notin U(\mu; t)$, then $\mu(x + y, q) \land \mu(y, q) \geq t > \mu(x, q)$, is a contradiction. Therefore by Lemma 3.5, $U(\mu; t)$ is a right $k$-ideal of $R$.

Conversely, if there exists $x, y \in R, q \in Q$ such that $\mu(x, q) < \mu(x + y, q) \land \mu(y, q)$, then $x + y, y \in U(\mu; t)$ and $x \notin U(\mu; t)$, where $t = \mu(x + y, q) \land \mu(y, q)$ which is a contradiction. Therefore by Lemma 3.5, $\mu$ is a $Q$-fuzzy right $k$-ideal of $R$.

Corollary 3.12 A right (left) ideal $I$ in $R$ is a right (left) $k$-ideal if and only if $\chi_I$ is a $Q$-fuzzy right (left) $k$-ideal of a semiring $R$.

Lemma 3.13 A $Q$-fuzzy set $\lambda$ of a semiring $R$ is an anti $Q$-fuzzy right $k$-ideal if and only if $L(\lambda; t)$ is a right $k$-ideal in a semiring $R$ for all $t \in [0, 1]$ whenever nonempty.

Proof. Let $\lambda$ be an anti $Q$-fuzzy right $k$-ideal of a semiring $R$. Let $t \in [0, 1]$. If there exists $x, y \in R, q \in Q$ such that $x + y, y \in L(\lambda; t)$ and $x \notin L(\lambda; t)$, then $\lambda(x + y, q) \lor \lambda(y, q) \leq t < \lambda(x, q)$, is a contradiction. Therefore by Lemma 3.5, $L(\lambda; t)$ is a right $k$-ideal in a semiring $R$.

Conversely, if there exists $x, y \in R, q \in Q$ such that $\lambda(x, q) > \lambda(x + y, q) \lor \lambda(y, q)$, then $x + y, y \in L(\lambda; t)$ and $x \notin L(\lambda; t)$, where $t = \lambda(x + y, q) \lor \lambda(y, q)$ which is a contradiction. Therefore by Lemma 3.5, $\lambda$ is an anti $Q$-fuzzy right $k$-ideal of a semiring $R$.

Theorem 3.14 An intuitionistic $Q$-fuzzy set $A = \{< x, q, \mu_A(x, q), \lambda_A(x, q) > | x \in R, q \in Q \}$ in $R$ is an intuitionistic $Q$-fuzzy right $k$-ideal in $R$ if and only if $U(\mu_A; t)$ is a right $k$-ideal in a semiring $R$ and $L(\lambda_A; t)$ is a right $k$-ideal in a semiring $R$ for all $t \in R$ whenever nonempty.

Proof. The proof follows from Lemma 3.5 and Lemma 3.11.
Corollary 3.15 An intuitionistic $Q$-fuzzy set $A = \{ < x, q, \mu_A(x, q), \lambda_A(x, q) > | x \in R, q \in Q \}$ in $R$ is an intuitionistic $Q$-fuzzy right $k$-ideal in $R$ if and only if $\mu_A$ is a $Q$-fuzzy right $k$-ideal in semiring $R$ and $\lambda_A$ is an anti $Q$-fuzzy right $k$-ideal in a semiring $R$.

Proof. The proof follows from Theorem 3.14 and Lemmas 3.11 and 3.13.

Corollary 3.16 An intuitionistic $Q$-fuzzy set $A = \{ < x, q, \mu_A(x, q), (1 - \mu_A)(x, q) > | x \in R, q \in Q \}$ in $R$ is an intuitionistic $Q$-fuzzy right $k$-ideal in $R$ if and only if $\mu_A$ is a $Q$-fuzzy right $k$-ideal in semiring $R$.

Proof. The proof follows from Theorem 3.14 and Corollary 3.15.

Corollary 3.17 An intuitionistic $Q$-fuzzy set $A = \{ < x, q, (1 - \lambda_A)(x, q), \lambda_A(x, q) > | x \in R, q \in Q \}$ in $R$ is an intuitionistic $Q$-fuzzy right $k$-ideal in $R$ if and only if $\lambda_A$ is an anti $Q$-fuzzy right $k$-ideal in a semiring $R$.

Proof. The proof follows from Theorem 3.14 and Corollary 3.15.

References


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