On Slightly $b$-Continuous Functions

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Abstract

The aim of this paper is to introduce a new set of properties of slightly $b$-continuous functions. Also the relations of slightly $b$-continuous functions with other weak forms of $b$-continuous functions have been investigated.

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1 Introduction

Andrijevic [1] introduced the notion of $b$-open sets in a topological space and obtained their various properties. El-Etik [2] introduced the same concept in the name of $\gamma$-open sets. El-Etik also introduced the concept of $\gamma$-continuous ($b$-continuous) functions with the aid of $b$-open sets. In 2004, Ekici and Caldas [3] introduced the notion of slightly $\gamma$-continuity (slightly $b$-continuity) which is a weakened form of $b$-continuity. In their paper, the authors have studied basic properties and preservation theorems of slightly $b$-continuous functions. The relationships of slightly $b$-continuity with other weaker forms of continuity have also been studied. In the present paper, a new set of conditions which characterize slightly $b$-continuous functions have been investigated. Also, the relation of slightly $b$-continuity with other weaker forms of $b$-continuity, viz. weakly $b$-continuity [4], somewhat $b$-continuity [5], almost $b$-continuity [6] and faintly $b$-continuity [7] have been studied.

2 Preliminary Notes

Throughout the present paper, $X$ and $Y$ are always topological spaces. Let $A$ be a subset of $X$. The interior and closure of a set are denoted by $\text{int}(A)$
and cl($A$), respectively. A subset of a topological space $X$ is said to be $b$-open \([1](\gamma$-open \([2])$ if $A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$. The complement of a $b$-open set is called $b$-closed \([1]$. The intersection of all $b$-closed sets of $X$ containing $A$ is called the $b$-closure \([1]$ of $A$ and is denoted by $b\text{cl}(A)$. A subset $B$ of $X$ is said to be a $b$-neighbourhood \([1]$ of a point $x \in X$ if there exists a $b$-open set containing $x$ and is contained in $A$. A subset $A$ of $X$ is said to be $\delta^*$-open \([8]$ if for each $x \in A$ there exists a clopen subset $G$ of $X$ such that $x \in G \subset A$. A subset $B$ of $X$ is said to be $\delta$-closed \([8]$ if $X \setminus B$ is $\delta$-open. The intersection of all $\delta^*$-closed sets of $X$ containing $A$ is called the $\delta^*$-closure of $A$ and is denoted by $\delta^*\text{cl}(A)$. A subset $A$ of $X$ is said to be $\theta$-open \([9]$ if every point of $A$ has an open neighbourhood whose closure is contained in $A$. A subset $A$ of $X$ is said to be regular open \([10]$ if $A = \text{int}(\text{cl}(A))$. The family of all $b$-open (resp. $b$-closed, clopen, $\delta^*$-open, $\delta$-closed, regular open) sets in $X$ is denoted by $BO(X)$ (resp. $BC(X)$, $CO(X)$, $BCO(X)$, $\delta^*O(X)$, $\delta^*C(X)$, $RO(X)$).

**Definition 2.1** A function $f : X \rightarrow Y$ is said to be almost $b$-continuous (briefly a.b.c.) \([6]$ if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in BO(X)$ containing $x$ such that $f(U) \subset V$.

**Definition 2.2** A function $f : X \rightarrow Y$ is said to be weakly $b$-continuous (briefly w.b.c.) \([4]$ if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there exists $U \in BO(X)$ containing $x$ such that $f(U) \subset \text{cl}(V)$.

**Definition 2.3** A function $f : X \rightarrow Y$ is said to be somewhat $b$-continuous (briefly s.w.b.c.) \([5]$ if for each open set $V$ in $Y$ and $f^{-1}(V) \neq \emptyset$ there exists $U \in BO(X)$ such $U \neq \emptyset$ and $U \subset f^{-1}(V)$.

**Definition 2.4** A function $f : X \rightarrow Y$ is said to be faintly $b$-continuous (briefly f.b.c.) \([7]$ if for each $x \in X$ and each $\theta$-open set $V$ in $Y$ containing $f(x)$, there exists $U \in BO(X)$ containing $x$ such that $f(U) \subset V$.

**Definition 2.5** A function $f : X \rightarrow Y$ is called slightly $\gamma$-continuous \([3]$ if for each $x \in X$ and each $V \in CO(X)$ containing $f(x)$, there exists a $U \in BO(X)$ containing $x$ such that $f(U) \subset V$.

In the present paper a slightly $\gamma$-continuous function will be termed as a slightly $b$-continuous function (briefly s.b.c.).

**Theorem 2.6** For a function $f : X \rightarrow Y$ the following are equivalent\([3]\):

(a) $f$ is s.b.c.;

(b) $f^{-1}(V) \in BO(X)$ for every $V \in CO(X)$;

(c) $f^{-1}(V) \in BC(X)$ for every $V \in CO(X)$;

(d) $f^{-1}(V) \in BCO(X)$ for every $V \in CO(X)$. 
3 Main Results

The following theorem gives a new set of conditions which characterize slightly b-continuous functions.

**Theorem 3.1** For a function $f : X \rightarrow Y$ the following are equivalent:

(a) $f$ is s.b.c.;

(b) $f^{-1}(V) \in BO(X)$ for every $\delta^*\text{-open } V$ in $Y$;

(c) $f^{-1}(V) \in BC(X)$ for every $\delta^*\text{-closed } V$ in $Y$;

(d) $f(bcl(A)) \subset \delta^*\text{-cl}(f(A))$ for every subset $A$ of $X$;

(e) $bcl(f^{-1}(B)) \subset f^{-1}(\delta^*\text{-cl}(B))$ for every subset $B$ of $Y$.

**Proof.** (a)$\Rightarrow$(b): Let $V$ be a $\delta^*$-open set in $Y$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$. The $\delta^*$-openness of $V$ gives a $U \in CO(Y)$ such that $f(x) \in U \subset V$. This implies that $x \in f^{-1}(U) \subset f^{-1}(V)$. Since $f$ is s.b.c., from Theorem 2.5., we have, $f^{-1}(U) \in BO(X)$. Hence $f^{-1}(V)$ is a $b$-neighbourhood of each of its points. Consequently, $f^{-1}(V) \in BO(X)$.

(b)$\Rightarrow$(c): It is obvious from the fact that the complement of a $\delta^*$-closed set is $\delta^*$-open.

(c)$\Rightarrow$(d): Let $A$ be a subset of $X$. We have, $\delta^*\text{-cl}(f(A)) = \cap\{F : f(A) \subset F, F \in \delta^*C(Y)\}$ is a $\delta^*$-closed set in $Y$. Thus $A \subset f^{-1}(\delta^*\text{-cl}(f(A))) = \cap\{f^{-1}(F) : f(A) \subset F, F \in \delta^*C(Y)\} \in BO(X)$. Thus, we obtain $bcl(A) \subset f^{-1}(\delta^*\text{-cl}(f(A)))$. Hence, $f(bcl(A)) \subset \delta^*\text{-cl}(f(A))$.

(d)$\Rightarrow$(e): Let $B$ be a subset of $Y$. We have $f(bcl(f^{-1}(B))) \subset \delta^*\text{-cl}(f(f^{-1}(B))) \subset \delta^*\text{-cl}(B)$ and hence, we obtain, $bcl(f^{-1}(B)) \subset f^{-1}(\delta^*\text{-cl}(B))$.

(e)$\Rightarrow$(a): Let $V$ be a clopen set in $Y$. Then $V$ is $\delta^*$-closed in $Y$. Thus $bcl(f^{-1}(B)) \subset f^{-1}(\delta^*\text{-cl}(B)) = f^{-1}(B)$. Therefore, $f^{-1}(B)$ is closed. Hence, by Theorem 2.6, we obtain $f$ is s.b.c.

**Theorem 3.2** If a function $f : X \rightarrow Y$ is w.b.c. then, $f$ is s.b.c.

**Proof.** Let $x \in X$ and let $V$ be a clopen set in $Y$ containing $f(x)$. Therefore, by weakly $b$-continuity of $f$, there exists $U \in BO(X)$ containing $x$ such that $f(U) \subset cl(V)$. Since, $x \in X$ is arbitrary, hence, $f$ is s.b.c.

**Remark 3.3** The converse of the above result is, however, far from true as shown by the following example.
Example 3.4 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the identity function $i : (X, \tau) \to (Y, \sigma)$ is s.b.c. but not w.b.c. at $b \in X$.

Remark 3.5 From definition, it is clear that every a.b.c. is w.b.c. and hence s.b.c. The converse is clearly false as shown by Example 3.4.

Definition 3.6 A space $X$ is said to be extremally disconnected [10] if closure of every open set is open in $X$.

Theorem 3.7 If a function $f : X \to Y$ is f.b.c. then, $f$ is s.b.c.

Proof. The result is obvious from the fact that every clopen set is $\theta$-open.

Remark 3.8 The converse of the above result is however, in general, not true as shown by the following example.

Example 3.9 Let $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$ and let $\sigma$ be the usual topology on $\mathbb{R}$. Then the identity function $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ is s.b.c. but not f.b.c. at all points of $\mathbb{R}$ except 0.

Theorem 3.10 A s.b.c. $f : X \to Y$ is f.b.c. if $Y$ is extremally disconnected.

Proof. Let $x \in X$ and let $V$ be a $\theta$-open set in $Y$ containing $f(x)$. Thus there exists an open set $W$ such that $f(x) \in \text{cl}(W) \subset V$. By extremally disconnectedness of $Y$, $\text{cl}(W)$ is open. Thus, $\text{cl}(W) \in CO(Y)$. Since, $f$ is s.b.c., therefore, there exists a $b$-open set $U$ containing $x$ such that $f(U) \subset \text{cl}(W) \subset V$. Since, $x \in X$ is arbitrary, therefore, $f$ is f.b.c.

Thus we have

Theorem 3.11 Let $f : X \to Y$ be a function, where, $Y$ is extremally disconnected. Then $f$ is f.b.c. if and only if $f$ is s.b.c.

Proof. It can be directly obtained by using Theorem 3.7 and Theorem 3.10.

Remark 3.12 Somewhat $b$-continuity and slightly $b$-continuity are independent of each other as shown by

Example 3.13 The function defined in Example 3.9 is s.b.c. but not sw.b.c. Again let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{b, c\}\}$. Then the identity function $i : (X, \tau) \to (Y, \sigma)$ is sw.b.c. but not s.b.c.

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References


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