Property for Graph of Some Commutative - Transitive Finite Rings

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Abstract

The main topic of this paper is to describe the structure of some graph commutative-transitive finite rings. It is shown that every such ring is a direct sum of an indecomposable noncommutative ring of prime power order, and a commutative ring. If for each $a, b, c \in R \setminus Z(R)$, $ab = ba$ and $bc = cb$ imply $ac = ca$, then the ring $R$ is said to be commutative-transitive. In this paper, we present graph of commutative-transitive rings. We show that a ring $R$ is commutative-transitive iff its commutative graph $\mu(R)$ is a union of complete graphs and present property for which the ring $M_n(R)$ is not commutative-transitive.

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1 Introduction

Definition 1.1 The ring $R$ is said to be commutative-transitive if for each $a, b, c \in R \setminus Z(R)$, $ab = ba$ and $bc = cb$ imply $ac = ca$ [1].

Definition 1.2 Let $R$ be a ring. Commutative graph $\mu(R)$ as set of vertices with non central elements is a ring the edges of which connect vertices $a \leftrightarrow b$ iff $a = b$ and distinct vertices $a, b$ in $\mu(R)$ adjacent iff $ab = ba$.

Definition 1.3 If $R$ is a ring and $\sigma : R \rightarrow R$ is an endomorphism, let $R[x; \sigma]$ denote the ring of polynomials over $R$ that is, all formal polynomials in $x$ with coefficients from $R$ with multiplication defined by $xr = \sigma(r)x$. [4]
Definition 1.4 Let G be a group and R a ring. Then; RG is defined as

\[ RG = \{ \sum_{g \in G} r_g g \mid r_g \in R \} \]

in which \( r_g = 0 \), except for some finite numbers. In RG, addition and multiplication are defined naturally and distributedly, respectively. RG is called a group ring on \( R \) [3].

Theorem 1.5 If \( R \) is a non-commutative ring with identity of order \( p^4 \), in which \( p \) is a prime number, then \( R \) is commutative-transitive.[1, Theorem 1-6]

Theorem 1.6 Pieress composition Theorem. Let \( e \) be idempotent in \( R \) ring. Then \( R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e) \). [3, page 318]

Theorem 1.7 Artin-veederborn theorem. Let \( R \) be any left semisimple ring. Then,

\[ R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k), \]

for suitable division rings \( D_1, \ldots, D_k \) and positive integers \( n_1, \ldots, n_k \). [3, page 35]

Theorem 1.8 Let \( F \) be a field and a nonevidend endomorphism of \( F \). Then \( R = \frac{F[X, \sigma]}{(x^2)} \) is commutative-transitive.[1, 2-2]

Theorem 1.9 Let \( R \) be a left artinian ring and \( n \in \mathbb{N} \), and \( F \) is a field such that \( \frac{R}{J(R)} \simeq M_n(F) \). Thus, local ring \( S \) exist such that \( R \simeq M_n(S) \).

2 The graph of some Commutative-transitive finite rings

Theorem 2.1 The ring \( R \) is commutative-transitive iff commutative graph \( \mu(R) \) is a union of complete graphs.

**Proof.** First, we show the following conditions are equivalent for ring \( R \):

a) \( R \) is commutative- transitive.

b) For each \( x, y \in R \setminus Z(R) \), if \( xy = yx \), then \( C(x) = C(y) \).

c) The centralizers of all non central elements of \( R \) are commutative.

\((a \rightarrow b)\) take \( x, y \in R \setminus Z(R) \) such that \( xy = yx \). If \( a \in C(x) \), then \( ax = xa \). Since \( xy = yx \) and \( R \) is commutative - transitive, \( ay = ya \). Then \( a \in C(y) \), resulting in the fact \( C(x) \subseteq C(y) \). Symmetrically \( C(y) \subseteq C(x) \). Then, \( C(x) = C(y) \).

\((b \rightarrow c)\) Suppose that \( x \in R \setminus Z(R) \). We show that if \( y, z \in C(x) \) then \( yz = zy \). If \( x \) and \( y \) are central, together they commutative. Then, take \( y, z \notin Z(R) \).
Since \( y, z \in C(x) \) then \( xy = yx \) and \( xz = zx \). So by condition (b) \( C(x) = C(y) \) and \( C(x) = C(z) \). Therefore, \( C(y) = C(z) \) and \( yz = yz \). \((c \to a)\) Let \( x, y, z \in R \setminus Z(R) \) such that \( xy = yx \) and \( yz = zy \) then, \( x, z \in C(y) \). By (c), \( C(y) \) is commutative, so \( xz = zx \).

**Proposition 2.2** The commutative graph of \( UT_2(F) \) has \(|F| + 1 \) unconected elements.

**proof:** If we sketch diagram of graph, the proof is completed.

\[
R = \left\{ \begin{bmatrix} a & b \\ 0 & a^2 \end{bmatrix} \mid a, b \in F_4 \right\},
\]

**Example 2.3** Let the following rings have 16 elements.

\[
S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} \mid a, b, c, d \in F_2 \right\}.
\]

Therefore, based on the above theorem, they are commutative-transitive and the commutative graph of \( R \) is the union of one \( K_6 \) graph and four \( K_2 \) graphs and the commutative graph of \( S \) is the union of three \( K_4 \) graphs.

**Note 1.** We have a single non commutative ring of \( p^3 \) order with identity of the following ring

\[
R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in Z_2 \right\}.
\]

That is also commutative-transitive [2]

**Example 2.4** Let \( F \) be a field and \( A = F\langle x_1, x_2, y_1, y_2 \rangle \), \( F \)-algebra generated by \( x_1, x_2, y_1 \) and \( y_4 \) such that \( x_i x_j = x_j x_i \) and \( y_i y_j = y_j y_i \) for \( i, j = 1, 2 \). \( x_i y_j - y_j x_i = 0 \) and \( x_i y_j - y_j x_i = 1 \) for \( i \neq j \). Clearly, \( x_1, x_2 \) and \( y_2 \) are three noncenteral elements for which \( x_2, y_2 \in C(x_1) \), but \( x_2 y_2 \neq y_2 x_2 \). Then, \( A \) is not commutative-transitive.

**Example 2.5** The ring

\[
R = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \mid a, b, c \in Z_2 \right\}.
\]

is commutative-transitive.

A graph transitive ring \( R \) is union of one graph \( K_3 \) and 4 graph \( K_1 \).
**Theorem 2.6** Let $R$ be ring with unity.

a) The ring $M_2(R)$ is commutative-transitive if and if $R$ is transitive domain.

b) The ring $M_n(R)$ is not commutative-transitive for $n \geq 3$.

**proof:** a) take $x, y, z \in M_2(R) \setminus Z(M_2(R))$, such that $yz = zy$ and $xy = yx$. We show that $xz = zx$. Consider

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}.$$

Subtracting multiples of unit $x, y, z$, suppose that $x_1 = y_1 = z_1 = 0$. From relation $xy = yx$. It is, therefore,

$$x_2y_3 = y_2x_3 \quad (1)$$

$$x_2y_4 = y_2x_4 \quad (2)$$

$$x_4y_3 = y_3x_3 \quad (3)$$

and relation $yz = zy$. It is, therefore,

$$z_2y_3 = y_2z_3 \quad (4)$$

$$z_2y_4 = y_2z_4 \quad (5)$$

$$z_4y_3 = y_4z_3 \quad (6)$$

Two sides (1) of right multiply $z_3$, so $x_2y_3z_3 = y_2x_3z_3$. $R$ is commutative and (4), then $x_2y_3z_3 = y_2x_3z_3 = x_3y_2z_3 = x_3z_2y_3$. Since $R$ is domain, $y$ is removed from two sides. Then

$$x_2z_3 = x_3z_2 \quad (7)$$

Two sides of (1) are right multiplied by $z_3$. So $x_2y_4z_3 = y_2x_4z_3$. $R$ is commutative, then

$$x_2y_4z_3 = y_2x_4z_3 = y_2z_4z_3 = x_4z_3z_4 = z_4y_4x_4.$$ 

Since $R$ is domain, $y_4$ is removed from two sides. Then

$$x_2z_4 = z_2x_4 \quad (8)$$

If two sides of (3) are right multiplied by $z_3$, with $R$ being commutative and (6), we have

$$x_4y_3z_3 = y_4x_3z_3 = y_4z_3z_3 = z_4y_3x_3.$$ 

Since $R$ is a domain, $y_3$ is removed from two sides. then

$$x_4z_3 = z_4x_3 \quad (9)$$
Therefore, it is followed from (7), (8) and (9) that $xz = zx$. So $M_2(R)$ is commutative-transitive.

We assume $R$ has zero divisor. So, for $a, b \in R \setminus \{0\}$, we have $ab = 0$, which can imply

$$x = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $xy = yx$ and $yz = zy$, but $xz \neq zx$ with commutative-transitive ring $M_n(R)$ has a contradiction. Similarly, if for some $a, b \in R, ab \neq ba$, then,

$$x = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$$

So $xy = yx$ and $yz = zy$, but $xz \neq zx$ which is in contradiction with commutative-transitive ring $M_n(R)$.

b) Take $x = E_{11}$, $y = E_{11} + E_{22}$ and $z = E_{21}$. Then, $x, y, z \in M_n(R) \setminus Z(M_n(R))$ and $yz = zy$ but $zx \neq zx$.

References


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