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On the Effect of Premia and Penalties on Optimal Portfolio Choice

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Abstract

In a standard portfolio choice between a risky and a safe asset, we study the effect of imposing premia and penalties conditional on the realized return of the portfolio meeting a given threshold. We show that thresholds set at "intermediate levels" have the effect to increase the optimal share of the safe asset, while very low and very high thresholds may induce larger shares of the risky investment if a condition on the curvature of the utility function holds.

Keywords: Portfolio, Premium, Risk Aversion

1 Introduction

In many economic and financial instances, the choice between a risky and a safe asset may involve premia and penalties that are conditional on the *ex post* realization of the portfolio. Examples include remuneration schemes that involve incentives based on realized sales, or stock option plans for CEOs sensitive to reputation or career concerns, see [1],[2]). Such premia and penalties may have an effect on the way agents optimally allocate resources (time, money, effort) between assets characterized by different degrees of risk. In this note we investigate this issue in the framework of a stylized model of optimal portfolio choice with one safe and one risky asset, and in which investors receive a bonus if the realized return of the portfolio exceeds a given threshold, and a penalty if the realized return falls short of the threshold. We compare the optimal share of risky asset with and without the threshold (that is, in a standard portfolio problem), to find that the threshold effect is to decrease the share of the risky asset as long as it is set not too far from the expected return of the optimal portfolio in absence of thresholds. When the threshold is set at high levels that are still reached (in the good state) by means of very risky portfolios, the choice of the investor may involve a larger share of risky asset. When the threshold is set at a level which is met in neither or both states of the world, the effect of the threshold on the share of risky asset is ambiguous, and ultimately depends on the curvature of the utility function.

2 The Model

Consider the traditional portfolio choice faced by a risk averse agent who allocates a given amount of money m between two assets, one safe with rate of return 1, and one risky, with per unit return given by $(1+\delta_H)>1$ with probability p and $(1-\delta_L)<1$ with probability $(1-p)$. We denote by $\alpha\in[0,1]$ the share of m that is invested in the risky asset. The random return from the α-portfolio is given by: $m_H(\alpha)\equiv m(1+\alpha\delta_H)$ with probability p and $m_L(\alpha)\equiv m(1-(\alpha\delta_L))$ with probability $(1-p)$. The expected utility from the α-portfolio is:

$$Eu(\alpha)=\left[(1-p)u(m_L(\alpha))+pu(m_H(\alpha))\right]. \tag{1}$$

where the Bernoulli utility function u is assumed increasing and strictly concave.

3 Portfolio Choice Without Bonus and Penalty

The optimal portfolio α^* solves the following maximization problem:

$$\max_{(\alpha\in[0,1])} Eu(\alpha) \tag{2}$$

Under mild assumptions on the Bernoulli utility function u we can rule out corner solutions with $\alpha=1$ and $\alpha=0$. Assuming internal solutions, we obtain the following first order conditions:

$$\frac{p\delta_H}{(1-p)\delta_L}=\frac{u'_L}{u'_H}, \tag{3}$$

where we have used the following notation:

$$u'_H\equiv u'(m_H(\alpha^*));\ u'_L\equiv u'(m_L(\alpha^*)). \tag{4}$$

4 Portfolio Choice with Bonus and Penalty

Consider now the case in which the investor receives a bonus $B > 0$ if the realized return of portfolio exceeds some given threshold $\bar{m}$, and a penalty $P < 0$ if the realized return of the portfolio falls short of $\bar{m}$.

We can define the following three regions for the choice variable (α), according to the realized portfolio return:

$$\Omega_1 = \{(\alpha) : m_H(\alpha) < \bar{m}\}; \tag{5}$$
$$\Omega_2 = \{(\alpha) : m_H(\alpha) \geq \bar{m},\ m_L(\alpha) < \bar{m}\}; \tag{6}$$
$$\Omega_3 = \{(\alpha) : m_L(\alpha) \geq \bar{m}\}. \tag{7}$$

Region Ω_1 is such that the treshold is never met; region Ω_2 is such that the treshold is met only in the good state of the world; region Ω_3 is such that the treshold is met in both states of the world.

The expected utility associated with these regions are:

$$E_1\bar{U}(\alpha) = (1-p)u\left(m_L(\alpha) + P\right) + pu\left(m_H(\alpha) + P\right) \tag{8}$$
$$E_2\bar{U}(\alpha) = (1-p)u\left(m_L(\alpha) + P\right) + pu\left(m_H(\alpha) + B\right) \tag{9}$$
$$E_3\bar{U}(\alpha) = (1-p)u\left(m_L(\alpha) + B\right) + pu\left(m_H(\alpha) + B\right) \tag{10}$$

Note that the maximization problem leading to an optimal portfolio choice is now discontinuous at all points on the frontier of the above regions. The next propositions establish conditions on the threshold level $\bar{m}$ under which the presence of bonuses and penalties have the effect of increasing the share of investment devoted to the safe asset.

Proposition 4.1 *Let $\bar{m} \in [m_L(\alpha^*), m_H(\alpha^*)]$. Let (α') solve problem (3). Then $\alpha' < \alpha^*$.*

Proof. When $\bar{m}$ is in the region $[m_L(\alpha^*), m_H(\alpha^*)]$, any increase of α above α^* has the effect of leaving the portfolio in the Ω_2 region. We can therefore check whether $\alpha' > \alpha^*$ can possibly satisfy the FOC of the modified portfolio problem:

$$\frac{p\delta_H}{(1-p)\delta_L} = \frac{u'(m(1-\delta_L\alpha') + P)}{u'(m(1+\delta_H\alpha') + B)} \tag{11}$$

Given that $u'' < 0$, $P < 0$ and $B > 0$, and given that α^* satisfies the FOC of the original problem:

$$\frac{1}{2}u'_H\delta_H = \frac{1}{2}u'_L\delta_L, \tag{12}$$

we obtain that $\alpha' < \alpha^*$.

Let us now consider the case in which $\bar{m} \notin [m_L(\alpha^*), m_H(\alpha^*)]$.

Proposition 4.2 *Let $\bar{m} < m_L(\alpha^*)$. Then if u'' is increasing (that is, decreasing in absolute value), then $\alpha' < \alpha^*$.*

Proof. Consider the FOC of the modified problem in the region Ω_3, which is attained for small variations of α around α^*:

$$\frac{p\delta_H}{(1-p)\delta_L} = \frac{u'(m(1-\delta_L\alpha^{\prime)} + B)}{u'(m(1+\delta_H\alpha') + B)} \tag{13}$$

Given that u'' is assumed increasing (decreasing in absolute value), and given FOC (3), $u'(m(1-\delta_L\alpha^*) + B) > u'(m(1+\delta_H\alpha^*) + B)$. This implies that (13) is satisfied by a level α' below α^*. *A fortiori*, an increase in the risky asset is not profitable when it brings the bad state into the penalty region.

Note that the above proof implies that when u' is decreasing, condition (13) can be satisfied by increased shares of the risky asset above α^*.

Let us then turn to the case in which the threshold is larger than $m_H(\alpha^*)$. It is now possible that the optimal α' may exceed α^* so to take advantage of the bonus in the good state of the world. This happens when $\bar{m}$ is achievable by means of a large enough share of the risky asset, that is when $\bar{m} < (1+\delta_H)$. When this is not the case, things turn out to depend, again, on the curvature of the utility function, as we show in the next proposition.

Proposition 4.3 *Let $\bar{m} > m_H(\alpha^*)$. If $(1+\delta_H) > \bar{m}$ then the optimal level of α' may exceed α^*. If $(1+\delta_H) < \bar{m}$, then $\alpha' < \alpha^*$ when u'' is decreasing (that is, increasing in absolute value).*

Proof. When $(1+\delta_H) > \bar{m}$ then the share of the risky asset may increase with respect to α^* in order to benefit from the bonus B, which is still in reach for large enough α'. When $(1+\delta_H) > \bar{m}$, the proposition is proved along symmetric and opposite arguments used in proposition 4.2, noting that the bonus B is replaced by the penalty P in equation (13).

References

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