Inverse Joint Moments of Multivariate Random Variables

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Abstract

Let $X_1,\ldots, X_p$ be $p$ random variables with joint distribution function $F(X_1,\ldots, X_p)$, and joint moment generating function $M_{X_1,\ldots, X_p}(t_1,\ldots, t_p)$. In this paper, the author has proposed methods for deriving inverse joint moments of multivariate random variables based on the joint moment generating function (mgf) of $X_1,\ldots, X_p$. Two examples are given in the bivariate case for illustration.

Keywords: Inverse; moments; multivariate; generating, bivariate; joint; Gamma.

1. Introduction

Inverse moments of random variables appear in many practical applications [1]. They may be applied in Stein estimation and post-stratification [2,3], evaluating risks of estimators and powers of tests [4,5]. They also appear in certain problems of reliability and survival analysis [6]. This type of moments is also important in life testing [7], insurance and financial mathematics [8], complex systems [9], and other areas. Garcia and Palacios [10] needed to evaluate $E[(a + X_n)^{-\alpha}]$, $\alpha > 0$ with $X_n$ having the binomial distribution $B(n,0.5)$, in order to obtain couplings of random walks on n-dimensional cubes. The calculations of $E[(a + X_n)^{-\alpha}]$, $\alpha > 0$ are almost difficult to calculate, and their bounds and approximations were explored instead [3,7,11,12].

Cressie and others [13], showed that inverse moments are hidden in the mgf of the probability distribution, provided that the mgf exists. They proposed methods based on mgf for computing inverse moments of univariate distributions. Carrying out the integrations may not be easy and in many cases, integrations may not yield closed-form expressions.
Therefore, many authors looked for good approximations, or even numerical solutions to the given integrals, see [1,12,14-17]. In this paper, the authors derive new methods for computing inverse joint moments of nonnegative r.v.'s with the knowledge of the joint mgf provided that the mgf exists. These methods are further extended to random variables defined on the $p$ dimensional space.

2. Results

2.1. Joint Inverse Moments

Suppose that $X_1,\ldots,X_p$ is $p$ positive random variables with distribution function $F(X_1,\ldots,X_p)$, and joint moment generating function $M_{X_1,\ldots,X_p}(t_1,\ldots,t_p) = E(\exp(\sum_i^p t_i X_i))$. Since $\left(\prod_i^p X_i\right)^{-1} = \int_0^\infty \cdots \int_0^\infty \exp(-t_1 x_1 + \cdots + t_p x_p)) \, dt_1 \cdots dt_p$, then, the $1^{st}$ inverse joint moment of $X_1,\ldots,X_p$ is given by

$$E\left(\prod_i^p X_i \right)^{-1} = \int_0^\infty \cdots \int_0^\infty \left(\prod_i^p X_i\right)^{-1} \, dF(X_1,\ldots,X_p),$$

$$= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp(-t_1 x_1 + \cdots + t_p x_p)) \, dt_1 \cdots dt_p \, dF(X_1,\ldots,X_p),$$

$$= \int_0^\infty \cdots \int_0^\infty M_{X_1,\ldots,X_p}(t_1,\ldots,t_p) \, dt_1 \cdots dt_p,$$

where the interchange of the order of integration is subject to $E(\exp(\sum_i^p t_i X_i))$ is integrable over $\mathbb{R}^{+p}$. It is not easy to perform integeration analytically, specially in the multivariate case, however, equation (1) is a mathematical tool for evaluating inverse moment which can be solved numerically.

Generalizations of equation (1) may be the $k^{th}$ joint inverse moment of $X_1,\ldots,X_p$, and the $(k_1,\ldots,k_p)$ th joint inverse moment of $X_1,\ldots,X_p$ that can be given in a similar way as

$$E\left(\prod_i^p X_i^{k_i} \right)^{-1} = \frac{1}{(\Gamma(k_i))^p} \int_0^\infty \cdots \int_0^\infty \prod_i^p t_i^{k_i-1} \, M_{X_1,\ldots,X_p}(-t_1,\ldots,-t_p) \, dt_1 \cdots dt_p,$$

and

$$E\left(\prod_i^p X_i^{k_i} \right)^{-1} = \frac{1}{\prod_i^p \Gamma(k_i)} \int_0^\infty \cdots \int_0^\infty \prod_i^p t_i^{k_i-1} \, M_{X_1,\ldots,X_p}(-t_1,\ldots,-t_p) \, dt_1 \cdots dt_p.$$

The above findings can be further extended according to the univariate case proposed earlier by Cressie and others [13], if we let $X_1,\ldots,X_p$ be $p$ random variables with distribution function $F(X_1,\ldots,X_p)$, and $Y_1 = \text{sign}(x_1), \ldots, Y_p = \text{sign}(x_p)$ be another set of $p$ random variables, where $\text{sign}(z) = 1$ if $z \geq 0$, $-1$ if, $z < 0$, then the $1^{st}$, $k^{th}$, and $(k_1,\ldots,k_p)$ th joint inverse moments can be given as follows:
Inverse joint moments

\[
E \left[ \prod^p X_i \right]^{-1} = \int_0^\infty \cdots \int_0^\infty \lim_{{t_{{p+1}}} \to 0^-} \left( \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \right) M_{X_i \cdots X_p} \bigg|_{X_i \cdots X_p = (-t_1, \ldots, -t_p, t_{p+1}, \ldots, t_{2p})} \, dt_1 \cdots dt_p \tag{4}
\]

\[
E \left[ \prod^p X_i^k \right]^{-1} = \frac{1}{\Gamma(k)} \int_0^\infty \cdots \int_0^\infty \prod^p t_i^{k-1} \times \lim_{{t_{{p+1}}} \to 0^-} \left( \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \right) M_{X_i \cdots X_p} \bigg|_{X_i \cdots X_p = (-t_1, \ldots, -t_p, t_{p+1}, \ldots, t_{2p})} \, dt_1 \cdots dt_p \tag{5}
\]

and

\[
E \left[ \prod^p X_i^k \right]^{-1} = \frac{1}{\prod^p \Gamma(k_i)} \int_0^\infty \cdots \int_0^\infty \prod^p t_i^{k_i-1} \times \lim_{{t_{{p+1}}} \to 0^-} \left( \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \right) M_{X_i \cdots X_p} \bigg|_{X_i \cdots X_p = (-t_1, \ldots, -t_p, t_{p+1}, \ldots, t_{2p})} \, dt_1 \cdots dt_p \tag{6}
\]

with the restriction \( F_{X_j}(0^+) = F_{X_j}(0) \), \( j = 1, 2, \ldots, p \). When \( X_1, \ldots, X_p \) are independent, random variables, we have that \( E \left[ \prod^p X_i \right]^{-1} = \prod^p E \left[ X_i^{-1} \right] \), \( E \left[ \prod^p X_i^k \right]^{-1} = \prod^p E \left[ X_i^k \right]^{-1} \) and \( E \left[ \prod^p X_i^k \right]^{-1} = \prod^p E \left[ X_i^k \right]^{-1} \).

### 2.2. Joint Inverse Central Moments

Suppose that \( X_1, \ldots, X_p \) is \( p \) positive random variables with distribution function \( F(X_1, \ldots, X_p) \), and joint moment generating function \( M_{X_1, \ldots, X_p}(t_1, \ldots, t_p) \). Cressie and others [13], showed that \( E \left[ (ax + b)^{-1} \right] = \int_0^\infty e^{-bt} M_X(-at) \, dt \), where \( X \) is a positive random variable having distribution function \( F(X) \), and mgf \( M_X(t) \) for all real numbers \( t \geq 0 \), and assuming that \( E(.) \) is finite. Thus

\[
E \left[ \prod^p (a_i X_i + b_i)^{-1} \right] = \int_0^\infty \cdots \int_0^\infty \prod^p (a_i X_i + b_i)^{-1} \, dF(X_1, \ldots, X_p).
\]

\[
= \int_0^\infty \cdots \int_0^\infty \exp(-\sum a_i t_i b_i) \left( \prod^p \exp(-\sum a_i t_i b_i) \right) \, dF(X_1, \ldots, X_p),
\]

\[
= \int_0^\infty \cdots \int_0^\infty \exp(-\sum a_i t_i b_i) \left( \prod^p \exp(-\sum a_i t_i b_i) \right) \, dF(X_1, \ldots, X_p),
\]

\[
= \int_0^\infty \cdots \int_0^\infty \exp(-\sum a_i t_i b_i) \, M_{X_1, \ldots, X_p}(-a_1 t_1, \ldots, -a_p t_p) \, dt_1 \cdots dt_p \tag{7}
\]

From (7), we can define the 1st joint inverse central moment of \( X_1, \ldots, X_p \) to be,

\[
E \left[ \prod^p (X_i - \mu_i)^{-1} \right] = \int_0^\infty \cdots \int_0^\infty \prod^p \mu_i^{-1} M_{X_1, \ldots, X_p}(-t_1, \ldots, -t_p) \, dt_1 \cdots dt_p \tag{8}
\]
Equation (8), can be generalized for the \( k^{th} \) joint central inverse moment of \( X_1,...,X_p \) to be given as

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-k_i}\right] = \frac{1}{\Gamma(k)} \int \cdots \int \prod_{i=1}^{p}t_i^{k_i-1} e^{-\sum_{i=1}^{p}\mu_i X_i} (-t_1,...,-t_p) \, dt_1 \cdots dt_p, \tag{9}
\]

or even extended to the \((k_1,...,k_p)\)th joint inverse central moments of \( X_1,...,X_p \) to be as

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-k_i}\right] = \frac{1}{\prod_i \Gamma(k_i)} \int \cdots \int \prod_{i=1}^{p}t_i^{k_i-1} e^{-\sum_{i=1}^{p}\mu_i X_i} (-t_1,...,-t_p) \, dt_1 \cdots dt_p. \tag{10}
\]

With the restriction \( F_{X_j}(0^+) = F_{X_j}(0), \ j = 1,2,...,p \). As before, if we let \( X_1,...,X_p \) be \( p \) random variables with distribution function \( F(X_1,...,X_p) \), and \( Y_j = \text{sign}(x_j),...,Y_p = \text{sign}(x_p) \) be another set of \( p \) random variables, where \( \text{sign}(z) = 1 \) if \( z \geq 0 \), \( = -1 \) if, \( z < 0 \), then the 1st, \( k^{th} \), and \((k_1,...,k_p)\)th joint inverse moments can be given as follows:

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-1}\right] = \int \cdots \int e^{-\sum_{i=1}^{p}\mu_i X_i} \, dt_1 \cdots dt_p, \tag{11}
\]

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-k_i}\right] = \frac{1}{\prod_i \Gamma(k_i)} \int \cdots \int \prod_{i=1}^{p}t_i^{k_i-1} e^{-\sum_{i=1}^{p}\mu_i X_i} (-t_1,...,-t_p) \, dt_1 \cdots dt_p, \tag{12}
\]

and

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-k_i}\right] = \frac{1}{\prod_i \Gamma(k_i)} \int \cdots \int \prod_{i=1}^{p}t_i^{k_i-1} e^{-\sum_{i=1}^{p}\mu_i X_i} (-t_1,...,-t_p) \, dt_1 \cdots dt_p. \tag{13}
\]

When \( X_1,...,X_p \) are independent, we have that

\[
E\left[\prod_{i=1}^{p}(X_i - \mu_i)^{-k_i}\right] = \prod_i \left[ E\left[(X_i - \mu_i)^{-k_i}\right] \right].
\]

3. Examples

In this section, two examples on the bivariate case will be illustrated using the proposed methods.
Example 1: Let \( X, Y \) have a Kibble and Moran bivariate gamma distribution, \( BG(\alpha, \beta) \) with shape parameter \( \alpha \) and scale parameter \( \beta = (\beta_1, \beta_2, \beta_3) \) \cite{18}, then the Laplace transform of \( X, Y \), is given by

\[
L(t_1, t_2) = (1 + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_2^2)^{-\alpha}, \quad t_1 > 0, \ t_2 > 0,
\]

where \( \alpha, \beta_1, \beta_2, \beta_3 > 0 \), and \( \beta_1 \beta_2 - \beta_3 = 0 \). Since for \( X, Y \) nonnegative r.v.'s, \( L(t_1, t_2) = M_{XY}(-t_1, -t_2) \), therefore, by (1) we have that

\[
E((XY)^{-1}) = \frac{1}{\beta_1 \beta_2 (\alpha - 1)^2} \left[ 2 F_1 \left( \frac{1, 1}{\alpha - 1}, \frac{\beta_1}{\beta_1 \beta_2} \right) + \frac{\pi (\alpha - 1)(\beta_1 \beta_2)^{-\alpha+1}}{1 - \beta_1 \beta_2} \right] \csc(\pi \alpha)
\]

where \( 2 F_1(a, b, c, z) \) is the Hypergeometric function, and

\[
2 F_1(a, b, c, z) = 1 + \frac{ab z}{c} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} + \cdots = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad |z| < 1.
\]

From (2) and (3) we have

\[
E((XY)^{-1}) = \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)} \Gamma(k) 0_k t_2^{-k} \left( \beta_1 + \beta_2 t_2 \right)^{-k} (1 + \beta_2 t_2)^{-\alpha+k} dt_2, \quad k > 0.
\]

and

\[
E(X^{-k} Y^{-r}) = \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)} \Gamma(k) 0_k t_2^{-k} \left( \beta_1 + \beta_2 t_2 \right)^{-k} (1 + \beta_2 t_2)^{-\alpha+k} dt_2, \quad k > 0, r > 0.
\]

When \( \beta_1 \beta_2 = \beta_3 \), The case \( X, Y \) are independent, equations (6), (7) and (8) give

\[
E((XY)^{-1}) = \frac{\beta_1 \beta_2}{(\alpha - 1)^2} \alpha > 1, \quad E((XY)^{-k}) = \frac{(\beta_1 \beta_2)^{-k}}{(\alpha - 1)\cdots(\alpha - k)^2}, \quad \alpha > k, k > 0, \quad \text{and}
\]

\[
E(X^{-k} Y^{-r}) = \frac{(\beta_1 \beta_2)^{-k}}{(\alpha - 1)\cdots(\alpha - k)^2}, \quad \alpha > \max(k, r).
\]

Example 2: Let \( X, Y \) have a bivariate weighted exponential distribution proposed by Al-Mutairi and other \cite{19}, \( (X, Y) \sim BWE(\lambda_1, \lambda_2, \lambda_3) \). The joint pdf of \( X, Y \) is

\[
f_{X,Y}(x, y) = \frac{2 \lambda_1 \lambda_2}{\lambda_3} e^{-\lambda_1 x} e^{-\lambda_2 y} (1 - e^{-\lambda_3 z}),
\]

where \( z = \min(x, y) \), and \( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \), \( \lambda_1, \lambda_2, \lambda_3 > 0 \). The moment generating function of \( X, Y \), is given by,

\[
M_{XY}(t_1, t_2) = \left( 1 - \frac{t_1}{\lambda_1} \right)^{-1} \left( 1 - \frac{t_2}{\lambda_2} \right)^{-1} \left( 1 - \frac{t_1 + t_2}{\lambda} \right)^{-1}.
\]

Then by (1), (2) and (3), we have that

\[
E((X_i X_2)^{-1}) = \lambda_1 \lambda_2 \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - \lambda} \right)^{-1} \left[ \pi^2 + (1 - \ln(\lambda_2))(\ln(\lambda_1) + \ln(\lambda - \lambda_1)) \right] + 2 \text{di}log\left( \frac{\lambda}{\lambda - \lambda_2} \right)
\]

\[
+ 2 \text{di}log\left( \frac{\lambda}{\lambda - \lambda_1} \right) - 2 \text{di}log(\lambda_1),
\]

where \( \text{di}log(z) = \int_{\frac{\ln(u)}{1-u}}^{z} \frac{1}{1-u} du \). From (7), and (8) we have

\[
E((XY)^{-k}) = \pi \csc(k \pi) \lambda_1 \lambda_2 \int_{0}^{\infty} t_2^{-1} \left( \frac{t_1 - t_2}{t_1 - t_2} \right) \left[ \frac{1}{(\lambda_1 + t_2) \lambda_2 + t_2) k^{-1} - \lambda_1 k^{-1} \lambda_2 k^{-1} - 1) dt_2, \quad k > 0.
\]

Invers joint moments
4. Discussion

Moment generating functions can describe distributions that are not easily defined. Although some distributions are inaccessible, moments and inverse moments can be derived through mgf’s. In this paper, the use of mgf’s for deriving inverse joint moments of multivariate random variables is shown. In many cases, performing integration on the joint mgf analytically is not easy and rather complex; nevertheless, the results of this paper can be of great interest for evaluating the integrals using approximations or even numerically. Further work is needed, especially for the inverse central moment case.

References


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