On the Geodetic Number of Line Graph

Venkanagouda M. Goudar
Sri Gouthama Research Center [Affiliated to Kuvempu University]
Department of Mathematics, Sri Siddhartha Institute of Technology
Tumkur, Karnataka, India
vmgouda@gmail.com

K. S. Ashalatha
Sri Gouthama Research Center [Affiliated to Kuvempu University]
Sri Siddhartha Institute of Technology, Tumkur, Karnataka, India
Department of Mathematics, Government First Grade College,
Gubbi, Tumkur, Karnataka, India
Eashu7kslatha@gmail.com

Venkatesha
Department of Mathematics, Kuvempu University
Shankarghatta, Shimoga, Karnataka, India

M. H. Muddebihal
Department of Mathematics, Gulbarga University
Gulbarga Karnataka, India

Abstract
For any graph $G(V, E)$, the line graph $L(G)$, whose vertices corresponds to the edges of $G$ and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent. For two vertices $u$ and $v$ of $G$ the set $I(u, v)$ consists of all vertices lying on a $u$-$v$ geodesic in $G$. If $S$ is a set of vertices of $G$ then $I(S)$ is
the union of all sets $I(u, v)$ for vertices $u$ and $v$ in $S$. The geodetic number $g(G)$ is the minimum cardinality among the subsets $S$ of $V(G)$ with $I(S) = V(G)$. In this paper we obtain the geodetic number of line graph of any graph. Also, obtain many bounds on geodetic number in terms of elements of $G$ and covering number of $G$.

Mathematics Subject Classification: 05C05, 05C12

Keywords. Cross product, Distance, Geodetic number, Line graph, Vertex covering number

1. Introduction

In this paper we follow the notations of [6]. All the graphs considered here are finite, non trivial, undirected and connected. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph $G$, respectively. For any graph $G(V, E)$, the Line graph $L(G)$ whose vertices correspond to the edges of $G$ and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$-$v$ path in $G$. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, rad $(G)$, and the maximum eccentricity is its diameter, diam $(G)$. A $u$-$v$ path of length $d(u, v)$ is called a $u$-$v$ geodesic. We define $I(u, v)$ to be the set (interval) of all vertices lying on some $u$-$v$ geodesic of $G$ and for a nonempty subset $S$ of $V(G)$,

$$I(S) = \bigcup_{u, v \in S} I(u, v).$$

A set $S$ of vertices of $G$ is called a geodetic set in $G$ if $I(S) = V(G)$ and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in $G$ is called the geodetic number $g(G)$.

Now we define geodetic number of line graph of a graph $G$. A set $S$ of vertices of $L(G) = H$ is called a geodetic set in $H$ if $I(S) = V(H)$ and a geodetic set of minimum cardinality is the geodetic number of $L(G)$ and is denoted by $g[L(G)]$. The Cartesian product (or direct product) $X \times Y$ of two sets $X$ and $Y$ is the set of all possible ordered pairs whose first component is a member of $X$ and whose second component is a member of $Y$. A vertex $v$ is an extreme vertex in a graph $G$, if the sub graph induced by its neighbors is complete. A vertex cover (edge cover) in a graph $G$ is a set of vertices (edges) that covers all edges (vertices) of $G$. The
minimum number of vertices (edges) in a *vertex cover* (edge cover) of $G$ is the vertex cover number $\alpha_v(G)$ (edge cover number $\alpha_e(G)$) of $G$.

For any undefined terms in this paper, see [5], [6].

2. Preliminary Results

**Theorem 2.1** [4] Every geodetic set of a graph contains its extreme vertices.

**Theorem 2.2** [4] If $G$ is a non trivial connected graph of order $n$ and diameter $d$, then $g(G) \leq n - d + 1$.

**Theorem 2.3** [4] Let $G$ be a connected graph of order at least 3. If $G$ contains a minimum geodetic set $S$ with a vertex $x$ such that every vertex of $G$ lies on some $x-w$ geodesic in $G$ for some $w \in S$, then $g(G) = g(G \times K_2)$.

**Propositions 1** The end edges of a tree $T$ are the extreme vertices of a line Graph $L(T)$ of $T$.

**Proposition 2** For any tree $T$ with order $n$ and diameter $d$, $L(T)$ and $T$ have the same value of $n - d$.

3. Main Results

**Theorem 3.1** For any tree $T$ with $k$ end edges, $g[L(T)] = k$.

**Proof.** Let $S$ be the set of all extreme vertices of a line graph $L(T)$ of a tree $T$. By the theorem 2.1, $g[L(T)] \geq |S|$. On the other hand, for an internal vertex $v$ of $T$, there exists $x,y$ of $T$ such that $v$ lies on the unique $x,y$ geodesic in $T$. The corresponding end edges of $T$ are the extreme vertices of $L(T)$. Thus $g[L(T)] \leq |S|$. Also every geodesic set $S'$ of $L(T)$ must contain $S$ which is the unique minimum geodesic set. Thus $|S| = |S'| = k$. Hence $g[L(T)] = k$.

**Corollary 3.1.1** For any path $P_n$ with $n$ vertices, $g[L(P_n)] = 2$.

**Proof.** Clearly the set of two end edges of a path $P_n$ is its unique geodesic set. From theorem 3.1, the result follows.

**Theorem 3.2** For any tree $T$ of order $n$ and diameter $d$, then $g(L(T)) \leq n - d + 1$.

**Proof.** Let $T$ be a non trivial connected graph of order $n$ and diameter $d$, let $u$ and $v$ be vertices of $L(T)$, for which $d(u,v) = d$. Let $u = v_0, v_1, \ldots, v_d = v$ be a $u-v$ path of length $d$. Now, let $S = V[L(T)] \setminus \{v_1, v_2, \ldots, v_d \setminus 1\}$. From the proposition 2, $I(S) = V[L(T)]$ and consequently $g(L(T)) \leq |S| = n - d + 1$. 
Theorem 3.3 For cycle $C_n$ of order $n \geq 3$, $g[L(C_n)] = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd}. \end{cases}$

**Proof.** The line graph $L(C_n)$ of a cycle $C_n$ is again a cycle. For the cycle $L(C_{2n})$, $n \geq 2$, the set of any two antipodal vertices is a geodetic set of $L(C_{2n})$. Also for $L(C_{2n+1}), n \geq 1$, no two vertices form a geodetic set, since there exists a 3 vertex geodetic set. Thus

$$g[L(C_n)] = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 3.4 If every non end vertex of a tree $T$ is adjacent to at least one end vertex, then $g[L(T)] \leq \left\lfloor n - \frac{k}{2} \right\rfloor$ where $k$ is number of end vertices in $T$.

**Proof.** If $diam(T) \leq 3$, then the result is obvious. Let $diam(T) > 3$ and $S' = \{v_1, v_2, ..., v_k\}$ be the set of all end vertices in $T$ with $|S'| = k$. Now without loss of generality, every end edge of $T$ are the extreme vertices of $L(T)$. Suppose $L(T)$ does not contain any edge vertex then $S = \{u_1, u_2, ..., u_i\}$, where $S \subseteq V[L(T)]$, forms a geodetic set of $L(T)$. Further if $L(T)$ contains at least one end vertex $w$, then the set $S \cup \{w\}$ forms a geodetic set $L(T)$. Therefore in all the cases, we obtain $|S \cup \{w\}| \leq \left\lfloor n - \frac{k}{2} \right\rfloor \Rightarrow g[L(T)] \leq \left\lfloor n - \frac{k}{2} \right\rfloor$.

Theorem 3.5 For any tree $T$, with $m$ edges, $g[L(T)] \leq m - \left\lfloor \frac{\alpha_1(T)}{2} \right\rfloor + 2$, where $\alpha_1(T)$ is an edge covering number.

**Proof.** Suppose $S' = \{e_1, e_2, ..., e_m\}$ be the set of all end edges in $T$. Then $S' \cup J$ where $J \subseteq E(T) - S'$, be the minimal set of edges which covers the vertices of $T$ and is not covered by $S'$, such that $|S' \cup J| = \alpha_1(T)$. Now without loss generality in $L(T)$, let $J = \{u_1, u_2, ..., u_i\} \subseteq V[L(T)]$ be the set of all vertices in $L(T)$ formed by the end edges in $T$ is the minimal geodetic set of $L(T)$. Clearly it follows that

$$g[L(T)] \leq |E(T)| - \left\lfloor \frac{|S' \cup J|}{2} \right\rfloor + 2 \Rightarrow g[L(T)] \leq m - \left\lfloor \frac{\alpha_1(T)}{2} \right\rfloor + 2.$$

Theorem 3.6 Let $G'$ be the graph obtained by adding an end edge $\{uv\}$ to a cycle $C_n = G$ with $u \in G$ and $v \notin G$ then $g[L(G')] = 3$ if $n$ is even.

**Proof.** Let $\{e_1, e_2, ..., e_n\}$ be a cycle with $n$ vertices which is even and let $G'$ be the graph obtained from $G = C_n$ by adding an end edge $\{uv\}$ such that $u \in G$ and $v \notin G$. By the definition of line graph, $L(G')$ has $\langle K_3 \rangle$ as an induced sub graph. Also the edge $\{uv\} = e_k$ becomes a vertex of $L(G')$ and it belongs to some geodetic set of $L(G')$. Hence $\{e_k, e_i, e_j\}$ are the vertices of $L(G')$ where $e_i, e_j$ are the edges incident on the antipodal vertex of $u$ in $G'$, and these vertices belongs to some geodetic set of $L(G')$. $L(G') = C_n \cup K_3$. Since $S = \{e_k, e_i, e_j\}$ is the minimum geodetic set. Therefore $g[L(G')] = 3$.

Theorem 3.7 $G'$ be the graph obtained by adding an end edge $\{uv\}$ to a cycle $C_n = G$ with $u \in G$ and $v \notin G$, then $g[L(G')] = 2$, if $n$ is odd.
Proof. Let \( \{e_1, e_2, ..., e_n, e_1\} \) be a cycle with \( n \) vertices which is odd and let \( G' \) be the graph obtained from \( G = C_n \) by adding an end edge \( \{uv\} \) such that \( u \in G \) and \( v \notin G \). By the definition of line graph, \( L(G') \) has \( K_3 \) as an induced sub graph, also the edge \( \{uv\} = e_k \) becomes a vertex of \( L(G') \). Let \( e_i = \{a, b\} \in G \) such that \( d(u, a) = d(u, b) \) in the graph \( L(G') \). Two elements subset of \( S = \{e_k, e_i\} \) of \( L(G') \) has the property that \( I(S) = V[L(G')] \). Thus \( g[L(G')] = 2 \).

**Theorem 3.8** Let \( G' \) be the graph obtained by adding end edge \( \{u_i, v_i\}, i = 1, 2, ..., n \) to each vertex of \( G = C_n \) such that \( u_i \in G, v_i \notin G \). Then \( g[L(G')] = g(G') = n \).

**Proof.** Let \( \{e_1, e_2, ..., e_n, e_1\} \) be a cycle with \( n \) vertices and \( G = C_n \). Let \( G' \) be the graph obtained by adding end edge \( \{u_i, v_i\}, i = 1, 2, ..., n \) to each vertex of \( G \) such that \( u_i \in G, v_i \notin G \). Clearly \( n \) be the number of end vertices of \( G' \). By the definition of line graph, \( L(G') \) have \( n \) copies of \( K_3 \) as an induced sub graph. The edges \( \{u_i, v_i\} = e_i \) for all \( i \), becomes \( n \) vertices of \( L(G') \) and those lies on geodetic set of \( L(G') \). Since they forms the extreme vertices of \( L(G') \), by theorem 2.1 \( g[L(G')] = g(G') = n \).

**Theorem 3.9** For any cycle \( C_n \), \( n \) is even, \( g[L(C_n)] = \frac{n}{\alpha_0(C_n)} \).

**Proof.** Let \( n \geq 3 \), is even be number of vertices and \( \alpha_0 \) is the vertex covering number of \( G \). We have \( L(C_n) = C_n \) and by theorem 3.3, \( g[L(C_n)] = 2 \). Also for even cycle, vertex covering number \( \alpha_0(C_n) = \frac{n}{2} \).

Hence \( g[L(C_n)] = 2 = \frac{n}{\alpha_0(C_n)} \).

**Theorem 3.10** For cycle, \( C_n \), \( n \) is odd, \( g[L(C_n)] = \frac{n}{\alpha_0(C_n)} + 1 \).

**Proof.** Let \( n \geq 3 \), is odd be number of vertices and \( \alpha_0 \) is the vertex covering number of \( C_n \). We have \( L(C_n) = C_n \) and by theorem 3.3, \( g[L(C_n)] = 3 \). Also for an odd cycle, vertex covering number \( \alpha_0(C_n) = \frac{n+1}{2} \).

Hence \( [L(C_n)] = 2 + 1 = \frac{n}{\alpha_0(C_n)} + 1 = \frac{n}{\alpha_0(C_n)} + 1 \).

**Theorem 3.11** For any integers \( m, n \geq 2 \), \( g[L(K_{m,n})] \leq mn - 1 \).

**Proof.** Let \( m + n \) and \( mn \) be the number of vertices and edges of the given graph \( K_{m,n} \) and \( d \) be the diameter. Since diameter of \( L(K_{m,n}) = 2 \), the number of vertices in \( L(K_{m,n}) \) is \( mn \). Hence by theorem 2.2 \( g(G) \leq n - d + 1 \). Now we have \( g[L(K_{m,n})] \leq mn - 2 + 1 \implies g[L(K_{m,n})] \leq mn - 1 \).

**Theorem 3.12** For any integer \( n \geq 4 \), \( g[L(K_n)] \leq \frac{(n+1)(n-2)}{2} \).

**Proof.** Let \( n \geq 4 \) be the vertices of the given graph \( K_n \) and \( d \) be the diameter. Since diameter of \( L(K_n) \) is 2 and the number of vertices in \( L(K_n) \) is \( \frac{n(n-1)}{2} \).
hence by theorem 2.2, \( g(G) \leq n - d + 1 \).
We have \( g[L(K_n)] = \frac{n(n-1)}{2} - 2 + 1 \).

\[ \Rightarrow g[L(K_n)] \leq \frac{n(n-1)}{2} - 1. \]

\[ \Rightarrow g[L(K_n)] \leq \frac{n^2 - n - 2}{2}. \]

\[ \Rightarrow g[L(K_n)] \leq \frac{(n + 1)(n - 2)}{2}. \]

**Theorem 3.13** For any path \( P_n \), \( g[L(P_n \times K_2)] = \begin{cases} 2, & \text{when } n = 2 \\ 3, & \text{when } n = 3 \\ 4, & \text{when } n > 3. \end{cases} \)

**Proof.** Let \( P_n \times K_2 \) be formed from two copies of \( G_1 \) and \( G_2 \) of \( P_n \). Then by

Theorem 2.3 \( g(P_n \times K_2) = g(P_n) \). Now \( L(P_n \times K_2) \) formed from two copies of \( G_1', G_2' \) of \( L(P_n) \). And let \( U = \{u_1, u_2, \ldots, u_{n-1}\} \in V(G_1'), W = \{w_1, w_2, \ldots, w_{n-1}\} \in V(G_2') \). We have the following cases.

**Case 1.** If \( n = 2 \), then by the definition of line graph, \( L(P_2 \times K_2) = P_2 \times K_2 \). By theorem 2.3 \( g[L(P_2 \times K_2)] = g(P_2) = 2 \).

**Case 2.** If \( n = 3 \), then \( L(P_3 \times K_2) \) is formed from two copies of \( P_2 \) clearly \( g[L(P_3 \times K_2)] = 3 \).

**Case 3.** Suppose \( n > 3 \). Let \( S \) be the geodetic set of \( L(P_n \times K_2) \). We claim that \( S \) contains two elements (end vertices) from each set \( \{u_1, u_{n-1}, w_1, w_{n-1}\} \). Since \( I(S) = V[L(P_n \times K_2)] \), it follows that \( g[L(P_n \times K_2)] \leq 4 \). It remains to show that if \( S' \) is a three element subset of \( V[L(P_n \times K_2)] \) then \( I(S') \neq V[L(P_n \times K_2)] \). First assume that \( S' \) is a subset \( U \) or \( W \), say the farmer. Then \( I(S') = S' \cup W \neq V \). Therefore, we may take that \( S' \cap U = \{u_i, u_j\} \) and \( S' \cap W = \{w_k\} \). Then \( I(S') = \{u_i, u_j\} \cup W \neq V[L(P_n \times K_2)] \).

**Theorem 3.14** For the wheel, \( W_n = k_1 + C_{n-1} \) \( (n \geq 6) \), \( n \) is even, \( g(L(W_n)) = \frac{n}{2} \).

**Proof.** Let \( W_n = k_1 + C_{n-1} \) \( (n \geq 6) \) with \( x \) the vertex of \( k_1 \) and \( V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\} \). \( E = \{e_1, e_2, \ldots, e_{n-1}\} \) be the internal edges of \( W_n \). Now, \( U = \{u_1, u_2, \ldots, u_j\} \) are the vertices formed from edges of \( C_{n-1} \) i.e \( U \subseteq V[L(W_n)] \). \( W = \{w_1, w_2, \ldots, w_j\} \) are the vertices of \( L(W_n) \) formed from internal edges of \( W_n \). i.e., \( W \subseteq V[L(W_n)] \). Now \( U \cup \{w_j\} \) forms a minimum geodetic set of \( L(W_n) \).

Clearly \( |U \cup \{w_j\}| = \frac{n}{2} \implies g[L(W_n)] = \frac{n}{2} \).

**Theorem 3.15** For the wheel \( W_n = k_1 + C_{n-1} \) \( (n \geq 6) \), \( n \) is odd, \( g[L(W_n)] = \frac{n+1}{2} \).
Proof. Let \( W_n = \{v_1, v_2, \ldots, v_{n-1}\} \), \( E = \{e_1, e_2, \ldots, e_{n-1}\} \) be the internal edges of \( W_n \). Now, \( U = \{u_1, u_2, \ldots, u_j\} \) are the vertices formed from edges of \( C_{n-1} \). i.e. \( U \subseteq V[L(W_n)] \).

\( W = \{w_1, w_2, \ldots, w_j\} \) are the vertices of \( L(W_n) \) formed from internal edges of \( W_n \). i.e., \( W \subseteq V[L(W_n)] \). Now \( U \cup \{w_j, w_{j-1}\} \) forms a minimum geodetic set of \( L(W_n) \), clearly \( |U \cup \{w_j, w_{j-1}\}| = \frac{n+1}{2} \Rightarrow g[L(W_n)] = \frac{n+1}{2} \).

References


Received: September, 2012