The Generalized K-Mittag-Leffler Function

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Abstract

In this paper author introduce generalized K- Mittag-Leffler function $GE_{k,a,b}(z)$ and prove some of its properties. Also deduce, Mittag-Leffler function introduced by [1], [2], [3], [4] and [6] are particular cases of generalized K-Mittag-Leffler function $GE_{k,a,b}(z)$.

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I Introduction

The K-Pochhammer symbol was introduce by [5] in the form,

$$ (x)_{n,k} = x(x+k)(x+2k) \ldots (x+(n-1)k), $$

where $x \in C, k \in R$ and $n \in N$. (1)

K-Gamma function was introduce by [5] in the form,

$$ \Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1} dt, x \in C, k \in R, Re(x) > 0 $$ (2)

Relation between classical Pochhammer symbol and K-Pochhammer symbol are given below (cf. [5]).
Proposition 1. Let \( \gamma \in \mathbb{C} \) and \( k, s \in \mathbb{R} \), then the following identity holds

\[
I_s(\gamma) = \left( \frac{s}{k} \right)^{\gamma-1} I_k \left( \frac{k \gamma}{s} \right),
\]

and particular case

\[
I_k(\gamma) = (k)^{\gamma-1} I_k \left( \frac{k}{k} \right).
\]

Proof. These identity will deduced by using equation (2).

Proposition 2. Let \( \gamma \in \mathbb{C}, k, s \in \mathbb{R} \) and \( n \in \mathbb{N} \), then the following identity holds

\[
(\gamma)_{n,q,s} = \left( \frac{s}{k} \right)^{nq} \left( \frac{ky}{s} \right)_{\gamma,n,q,k},
\]

and for particular case

\[
(\gamma)_{n,q,k} = (k)^{nq} \left( \frac{y}{k} \right)_{\gamma,n,q},
\]

Proof. Using equation (1), we get immediately result (5) and (6).

II Main results

In this section author introduce a new Mittag-Leffler type function, called as Generalized K-Mittag-Leffler function \( GE_{k,a,b}^{\gamma,q}(z) \), and consider some of its properties and particular cases.

Definition 1. Let \( k \in \mathbb{R}; \alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \) and \( q \in (0,1) \cup \mathbb{N} \), the Generalized K-Mittag-Leffler function denoted by \( GE_{k,a,b}^{\gamma,q}(z) \) and defined as

\[
GE_{k,a,b}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,q,k} z^n}{\Gamma_k(n\alpha + \beta)(n!)},
\]

where \( (\gamma)_{n,q,k} \) is the K-pochhammer symbol given by equation (1) and \( I_k(x) \) is the K-gamma function given by equation (2).

For some particular values of the parameters \( q, k, \alpha, \beta \) and \( \gamma \), we can obtained certain Mittag-Leffler functions, defined earlier:

**(a)** For \( q = 1 \), equation (7) reduces in K-Mittag-Leffler functions defined by [3].

\[
GE_{k,a,b}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k(n\alpha + \beta)(n!)} = E_{k,a,b}^{\gamma}(z).
\]

**(b)** For \( k = 1 \), equation (7) reduces in Mittag-Leffler functions defined by [1].
The generalized K-Mittag-Leffler function

\[ GE_{\gamma, q}^{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n \alpha + \beta)}{\Gamma(n \alpha + \beta)(n!)} (\gamma)_{nq} z^n \]

\[ GE_{\gamma, q}^{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n \alpha + \beta)}{\Gamma(n \alpha + \beta)(n!)} (\gamma)_{nq} z^n = E_{\alpha, \beta}^{\gamma, q}(z), \quad (9) \]

(c) For \( q = 1 \) and \( k = 1 \), equation (7) reduces in Mittag-Leffler functions defined by [6].

\[ GE_{1, 1}^{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n \alpha + 1)} = E_{\alpha, \beta}(z), \quad (10) \]

(d) For \( q = 1, k = 1, \) and \( \gamma = 1 \) equation (7) reduces Mittag-Leffler functions defined by [2].

\[ GE_{1, 1}^{1, 1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n \alpha + 1)} = E_{\alpha}(z) \quad (11) \]

(e) For \( q = 1, k = 1, \gamma = 1\) and \( \beta = 1 \) equation (7) reduces Mittag-Leffler functions defined by [4].

\[ GE_{1, 1}^{1, 1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n \alpha + 1)} = E_{\alpha}(z) \quad (12) \]

**Theorem 1.** The functional relation between Generalized K- Mittag-Leffler function and Mittag-Leffler function defined in equation (9), is given by

\[ GE_{k, \alpha, \beta}^{\gamma, q}(z) = k^{1-\beta} \frac{\gamma}{k} E_{\alpha, \beta}^{\gamma, q} \left( k^{-\frac{\alpha}{k}} z \right), \quad (13) \]

or the counterpart

\[ k^{1-\beta} GE_{k, \alpha, \beta}^{\gamma, q} \left( k^{\frac{\alpha}{k}} a z \right) = E_{\alpha, \beta}^{\gamma, q}(az), a \in \mathbb{R}. \quad (14) \]

Proof. From the definition of Generalized K- Mittag-Leffler function, equation (7),

\[ GE_{k, \alpha, \beta}^{\gamma, q}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n \alpha + \beta)}{\Gamma(n \alpha + \beta)(n!)} (\gamma)_{nq} z^n \]

using equations (4) and (6), we have

\[ GE_{k, \alpha, \beta}^{\gamma, q}(z) = k^{1-\beta} \sum_{n=0}^{\infty} \frac{\Gamma(n \alpha + \beta)}{\Gamma(n \alpha + \beta)(n!)} (\gamma)_{nq} \left( k^{\frac{\alpha}{k}} z \right)^n \]
Corollary 1. Put \( q = 1 \) in equation (13) and (14), we have the known result [3], Equation II.11 and II.12, Page 709.

Theorem 2. Let \( k \in R; \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \) and \( q \in (0,1) \cup N \), then

\[
GE_{k,\alpha,\beta}^{\gamma,q}(z) = k^{1-\beta} E_{\alpha,\beta}^{\gamma,q} \left( k^{\frac{\alpha}{\kappa}} z \right)
\]

Hence.

Corollary 2. Put \( q = 1 \) in equation (15), we have

\[
E_{k,\alpha,\beta}^{\gamma}(z) = \beta. E_{k,\alpha,\beta+k}^{\gamma}(z) + (az). \frac{d}{dz} E_{k,\alpha,\beta+k}^{\gamma}(z),
\]

which is the known result [3, Lemma 1, Page 709].

Theorem 3. Let \( k \in R; \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \) and \( q \in (0,1) \cup N \), then

\[
GE_{k,\alpha,\beta}^{\gamma,q}(z) - GE_{k,\alpha,\beta}^{\gamma-k,q}(z) = qk^q \left( \frac{\gamma}{k} \right)_{q-1} z. GE_{k,\alpha,\alpha+\beta}^{\gamma+kq,q}(z),
\]

Proof. Consider

\[
A \equiv GE_{k,\alpha,\beta}^{\gamma,q}(z) - GE_{k,\alpha,\beta}^{\gamma-k,q}(z),
\]

using the result given by equation (13) and the relation \( nq(x)_{nq-1} = (x)_{nq} - (x - 1)_{nq} \), we obtain
using definition given by equation (9), we have
\[ A \equiv k^{1-\frac{\beta}{k}} \sum_{n=1}^{\infty} \left( z k^{\left(q-\frac{\alpha}{k}\right)} \right)^{n} \frac{\binom{n}{q} \left(\frac{\gamma}{k}\right)}{\Gamma\left(\frac{\alpha}{k} + \frac{\beta}{k}\right)} (n!^q nq), \]
\[ A \equiv q k^{1-\frac{\beta}{k}} \sum_{n=0}^{\infty} \left( z k^{\left(q-\frac{\alpha}{k}\right)} \right)^{n+1} \frac{\binom{n+1}{q} \left(\frac{\gamma}{k}\right) (n+1!)^q (n+1)}{\Gamma\left(\frac{\alpha}{k} + \frac{\beta}{k}ight) (n+1)!}, \]
\[ A \equiv q k^{1+q-\frac{\alpha+\beta}{k}} z \sum_{n=0}^{\infty} \left( z k^{\left(q-\frac{\alpha}{k}\right)} \right)^{n} \frac{\binom{n}{q} \left(\frac{\gamma}{k} + q - 1\right)_{nq}}{\Gamma\left(\frac{\alpha}{k} + \frac{\beta}{k}\right) (n)!}, \]
using the relation \((\delta)_{n+j} = (\delta)_{j}(\delta + j)n\), we have
\[ A \equiv q k^{1+q-\frac{\alpha+\beta}{k}} z \sum_{n=0}^{\infty} \left( z k^{\left(q-\frac{\alpha}{k}\right)} \right)^{n} \frac{\binom{n}{q} \left(\frac{\gamma}{k} + q - 1\right)_{nq}}{\Gamma\left(\frac{\alpha}{k} + \frac{\beta}{k}\right) (n)!}, \]
using definition given by equation (9), we have
\[ A \equiv q k^{1+q-\frac{\alpha+\beta}{k}} \left( \frac{\gamma}{k} \right)_{q-1} z \sum_{n=0}^{\infty} \left( z k^{\left(q-\frac{\alpha}{k}\right)} \right)^{n} \frac{\binom{n}{q} \left(\frac{\gamma}{k} + q - 1\right)_{nq}}{\Gamma\left(\frac{\alpha}{k} + \frac{\beta}{k}\right) (n)!}, \]
using equation (13), we have
\[ A \equiv q k^{q} \left( \frac{\gamma}{k} \right)_{q-1} z. GE_{k,\alpha+\beta}^{Y+q-\frac{\alpha}{k}}(z). \]
Hence.

**Corollary 3.** The result given by [3], Lemma 2, Equation II.14, Page 710, is
\[ E_{k,\alpha,\beta}^{Y}(z) - E_{k,\alpha,\beta}^{Y-k}(z) = k^{1-\frac{\alpha}{k}} z E_{k,\alpha,\alpha+\beta}^{Y}(z), \]
which is wrong and the true result is given by putting \(q = 1\) in equation (16) we have the formula,
\[ E_{k,\alpha,\beta}^{Y}(z) - E_{k,\alpha,\beta}^{Y-k}(z) = k z E_{k,\alpha,\alpha+\beta}^{Y}(z) \]

**Theorem 4.** Let \(k \in R; \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, j \in N \) and \(q \in (0,1) \cup N\), then
\[ \left( \frac{d}{dz} \right)^{j} GE_{k,\alpha,\beta}^{Y+q}(z) = (\gamma)_{q} E_{k,\alpha,\alpha+\beta}^{Y+q}(z), \]
(17)

Proof. Consider left hand side and using equation (7)
\[
A \equiv \left( \frac{d}{dz} \right)^j \sum_{n=0}^{\infty} \frac{(y)_{nq,k}}{\Gamma_k(n\alpha + \beta)(n!)} z^n, \\
A \equiv \sum_{n=j}^{\infty} \frac{(y)_{nq,k}}{\Gamma_k(n\alpha + \beta)(n!)} (n-j)! z^{n-j}, \\
A \equiv \sum_{n=0}^{\infty} \frac{(y)_{(n+j)q,k}}{\Gamma_k(n\alpha + j\alpha + \beta)(n!)} z^n, \\
A \equiv (y)_{jq,k} E_k^{y+jq,k} G(z).
\]

Hence.

**Corollary 4.** Put \( q = 1 \) in equation (17), we have the known result [3], Lemma 3, Equation II.15, Page 710.

**Theorem 5.** Let \( k \in \mathbb{R}; \alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, \) and \( q \in (0,1) \cup \mathbb{N} \), then

\[
\sum_{n=0}^{\infty} (x+y)^n G_k^{nq+k,q} (-xy) = \sum_{r=0}^{\infty} \frac{(-xy)^r}{r!} \Gamma(qr+1) G_k^{rq+k,q} \left( \frac{x+y}{k} \right),
\]

Using equation (4) and (6), we have

\[
(nqk + k)_{rq,k} = k^rq \left( \frac{nqk+k}{k} \right)^r, \\
(nqk + k)_{rq,k} = k^rq (nq + 1)_rq, \\
(nqk + k)_{rq,k} = k^rq \frac{\Gamma(nq + 1 + rq)}{\Gamma(nq + 1)}, \\
(nqk + k)_{rq,k} = k^rq \frac{\Gamma(rq + 1 + nq)\Gamma(rq + 1)}{\Gamma(rq + 1)\Gamma(nq + 1)}, \\
(nqk + k)_{rq,k} = k^rq \frac{\Gamma(rq + 1)}{\Gamma(nq + 1)} (rq + 1)_{rq}^{nq}, \\
(nqk + k)_{rq,k} = k^rq \frac{\Gamma(rq + 1)}{\Gamma(nq + 1)} \left( \frac{rqk + k}{k} \right)^nq, \\
(nqk + k)_{rq,k} = k^rq \Gamma(rq + 1) \frac{\Gamma(nqk + k)}{\Gamma(nqk + k)},
\]

using equation (4) and (6), we have
The generalized K-Mittag-Leffler function \(2219\)

\[(nqk + k)_{rq,k} = k^{rq} \Gamma(rq + 1) \frac{(rqk + k)_{nq,k}}{\Gamma_k(nqk + k)},\]  \hspace{1cm} (19)

consider left hand side of (18) and using equation (7), we have

\[A \equiv \sum_{n=0}^{\infty} (x + y)^n \sum_{r=0}^{\infty} \frac{(nqk + k)_{rq,k} (-xy)^r}{r!} \Gamma_k(0, r + (n + 1)k)(r!)
\]

using equation (19), we have

\[A \equiv \sum_{n=0}^{\infty} (x + y)^n \sum_{r=0}^{\infty} k^{rq} \Gamma(rq + 1) \frac{(rqk + k)_{nq,k} (-xy)^r}{r!} \Gamma_k(nqk + k) \Gamma_k((n + 1)k)(r!)
\]

Hence.

References


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