An \(\alpha\)-Mellin Transform and Some of its Applications

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Abstract

In this work we present an \(\alpha\)-Mellin transformation and its inverse for each \(0 < \alpha \leq 1\), that reduces the usual transformation when \(\alpha = 1\). Furthermore we show that the usual properties of the Mellin transformation are the same for the \(\alpha\)-transformation, and it is therefore a generalization; moreover, considering the Black-Scholes equation we see that it is solvable through the \(\alpha\)-Mellin transformation to certains finance investment option.

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1 Introduction

1.1 Generalities on the Mellin Transformation

In this section we define the Mellin transformation and recall its main properties, we see the way that we can obtain the inverse Mellin transformation through the classical Fourier transformation [6].

Definition 1.1 Let \(f\) be a complex-valued function defined over \((0, \infty)\), locally integrable. To avoid further complication, we assume throughout that it is continuous in \((0, \infty)\). The Mellin transform is defined as

\[
\mathcal{M}(f, a) = F(s) = \int_0^\infty f(t)t^{s-1}dt.
\] (1)

In general, the integral does exist only for complex values of \(s = a + ib\), such that \(a_1 < a < a_2\), where \(a_1\) and \(a_2\) depend of the function \(f\).

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In the above definition $a_1$ and $a_2$ form the so-called strip of definition $S(a_1, a_2)$. In some cases, this strip may be extended to a half-plane ($a_1 = -\infty$ or $a_2 = \infty$) or to the whole complex $s$-plane ($a_1 = -\infty$ and $a_2 = \infty$).

Let us see now the relation between the Laplace transform and the Mellin one. By doing the change of variable $t = e^{-x}$, $dt = -e^{-x}$ in eq. (1), we have

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})e^{-sx}dx,$$

if we make $g(x) = f(e^{-x})$, the integral

$$\mathcal{L}(g, s) = \int_{-\infty}^{\infty} g(x)e^{-sx}dx,$$

can be seen as the two-side Laplace transform of $g$. This can be written symbolically as:

$$\mathcal{M}(f(t), s) = \mathcal{L}(f(e^{-x}), s), \quad s = a + ib.\quad (3)$$

Let us consider now, $s = a + 2\pi i\beta$ in eq. (2)

$$F(s) = \mathcal{M}(f(t)) = \int_{-\infty}^{\infty} f(e^{-x})e^{-x(a+2\pi i\beta)}dx = \int_{-\infty}^{\infty} f(e^{-x})e^{-ax}e^{-2\pi i\beta x}dx, \quad (4)$$

then we have

$$\mathcal{M}(f(t), s = a + 2\pi i\beta) = \mathcal{F}(f(e^{-x})e^{-ax}, \beta)\quad (5)$$

where $\mathcal{F}$ represents the Fourier transformation defined in the usual way

$$\hat{f}(\beta) = \mathcal{F}(f, \beta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\beta x}dx.\quad (6)$$

Thus, for every real number given by $a = \Re(s)$, such that $a \in S(a_1, a_2)$ in eq. (5), we can see that the Mellin transformation of a function $f$ can be expressed as a certain Fourier transformation.

A directed way to obtain the inverse Mellin transformation is via the inverse Fourier transformation. If $\hat{f}(\beta) = \mathcal{F}(f, \beta)$, then we recover $f$ through the inverse Fourier transformation, that is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\beta)e^{2\pi i\beta x}d\beta.\quad (7)$$

Using now equation (4) with $s = a + 2\pi \beta i$ and eq. (7) we have

$$f(e^{-x})e^{-ax} = \int_{-\infty}^{\infty} F(s)e^{2\pi i\beta x}d\beta,\quad (8)$$
and taking again the change of variable \( t = e^{-x} \), we obtain

\[ f(t)t^a = \int_{-\infty}^{\infty} F(s)t^{-2\pi i \beta} d\beta = t^{-a} \int_{-\infty}^{\infty} F(s)t^{-2\pi i \beta} d\beta. \]

From the latter we can see that the inverse Mellin transform is given by

\[ f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)t^{-s} ds. \quad (9) \]

On the other hand, if in eq. (6) we take \( \omega = 2\pi \beta \), i.e., in terms of the angular frequency

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = \mathcal{F}(f(x)), \]

and then

\[ f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x} d\omega, \]

and therefore

\[ \mathcal{M}(f(t), s = a + i\omega) = \mathcal{F}(f(e^{-x})e^{-ax}, \omega), \quad \forall a \in S(a_1, a_2). \]

2 \( \alpha \)-Mellin Transform

2.1 Fractional Fourier Transform

In ref. [5], [10] the authors introduced a new Fourier transformation (fractional) defined on the Lizorkin functions space, briefly:

Let \( S(\mathbb{R}) \) be the Schwartz space, i.e., the space of rapidly decreasing function on \( \mathbb{R} \), and consider also the dense subspace of the Schwartz space

\( V(\mathbb{R}) = \{ v \in S(\mathbb{R}) : v^{(n)}(0) = 0, n = 0, 1, 2, \ldots \} \);

now, the Lizorkin space of functions \( \Phi(\mathbb{R}) \) is defined as

\[ \Phi(\mathbb{R}) = \{ f \in S(\mathbb{R}) : \mathcal{F}(f) \in V(\mathbb{R}) \}. \quad (10) \]

In ref. [5], for every \( f \in \Phi(\mathbb{R}) \) and every \( 0 < \alpha \leq 1 \), the fractional Fourier transform (FFT) of order \( \alpha \) is defined by

\[ \hat{f}_\alpha(\omega) = \mathcal{F}_\alpha(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} e^{1/ \alpha} x \frac{1}{\alpha} d\omega, \quad (11) \]

and the inverse fractional Fourier transform of order \( \alpha \) as

\[ f(x) = \mathcal{F}_\alpha^{-1}(\hat{f}_\alpha(\omega)) = \frac{1}{2\pi \alpha} \int_{-\infty}^{\infty} \hat{f}_\alpha(\omega)e^{i\omega x} e^{-1/ \alpha} x \omega \frac{1}{\alpha} d\omega. \quad (12) \]

If, in eq. (11) and (12) we make \( \alpha = 1 \) we recover the usual Fourier transform and its inverse.
2.2 Fractional Mellin Transform

Taking into account the above section, if we define \( a priori \) a fractional Mellin transform \( \mathcal{M}_\alpha \) or an \( \alpha \)-Mellin transform, it must to satisfy that for every \( 0 < \alpha \leq 1 \),

\[
\mathcal{M}_\alpha(f(t), a + 2\pi i \beta) = \mathcal{F}_\alpha(f(e^{-x})e^{-ax}, \beta)
\]
or, in terms of the angular frequency, if we consider \( s = a + 2\pi i \beta = a + i\omega \), \( \omega = 2\pi \beta \)

\[
\mathcal{M}_\alpha(f(t), a + i\omega) = \mathcal{F}_\alpha(f(e^{-x})e^{-ax}, \omega^{\frac{1}{\alpha}}). 
\]

Thus, for every \( a \in S(a_1, a_2) \)

\[
\mathcal{M}_\alpha(f(t), a + i\omega) = \int_{-\infty}^{\infty} f(e^{-x})e^{-ax}e^{-i\omega^{\frac{1}{\alpha}}x} dx = \int_{-\infty}^{\infty} f(e^{-x})e^{-(a+i\omega^{\frac{1}{\alpha}})x} dx
\]

\[
= \int_{-\infty}^{\infty} f(e^{-x})(e^{-x})^{a+i\omega^{\frac{1}{\alpha}}} dx = \int_{0}^{\infty} f(t)t^{s_{\alpha}-1} dt,
\]

where we have used \( t = e^{-x} \), \( dx = -t^{-1}dt \) and for every \( 0 < \alpha \leq 1 \), \( s_{\alpha} = a+i\omega^{\frac{1}{\alpha}} \) with \( a \in S(a_1, a_2) \).

Considering now that the inverse Fourier \( \alpha \)-transform eq. (12) and using the definition of \( \mathcal{M}_\alpha(f(t), a + i\omega) \), we have that

\[
f(e^{-x})e^{-ax} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{M}_\alpha(f(t), a + i\omega)e^{i\omega^{\frac{1}{\alpha}}x} \omega^{\frac{1}{\alpha}} d\omega
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}_\alpha(f(e^{-x})e^{-ax}, \omega^{\frac{1}{\alpha}})e^{i\omega^{\frac{1}{\alpha}}x} \omega^{\frac{1}{\alpha}} d\omega
\]

from where

\[
f(t)t^{a} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha(f(t), s)t^{-i\omega^{\frac{1}{\alpha}}} ds.
\]

Therefore we have obtained

\[
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha(f(t), s)t^{-a}t^{-i\omega^{\frac{1}{\alpha}}} ds
\]

\[
= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha(f(t), s)t^{s_{\alpha}} ds,
\]

(13)

where \( t = e^{-x} \), and \( s_{\alpha} = a+i\omega^{\frac{1}{\alpha}} \), have been used. If we call \( F_{\alpha}(s) = \mathcal{M}_\alpha(f(t), s) \) then we have the inverse fractional Mellin transform

\[
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F_{\alpha}(s)t^{-s_{\alpha}} ds
\]

(14)

From the latter, the next definition can be given
**Definition 2.1** For every \( f \in \Phi(\mathbb{R}) \) and \( s_\alpha = a + i\omega_\alpha \) with \( a \in S(a_1, a_2) \) and \( 0 < \alpha \leq 1 \) we define the fractional Mellin transform \( \mathcal{M}_\alpha \) through

\[
\mathcal{M}_\alpha(f(t), s_\alpha) = F_\alpha(s) = \int_0^\infty f(t)t^{s_\alpha-1}dt
\]

whose inverted formula is

\[
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F_\alpha(s)t^{-s_\alpha}ds.
\]

### 2.3 Properties of the \( \alpha \)-Mellin Transform

Let \( f \) be a function belongs to the Lizorkin space \( \Phi(\mathbb{R}) \), \( k \in \mathbb{R} \). Then, of the Definition 2.1 we have the following statements:

**Proposition 2.2** Let \( f \) be a function belong of the Lizorkin space \( \Phi(\mathbb{R}) \), \( k \in \mathbb{R} \), then we have:

1. **Scaling property**
   \[
   \mathcal{M}_\alpha(kt, s_\alpha) = k^{-s_\alpha}\mathcal{M}_\alpha(f(t), s_\alpha);
   \]

2. **Multiplication by \( t^k \)**
   \[
   \mathcal{M}_\alpha(t^k f(t), s_\alpha) = \mathcal{M}_\alpha(f(t), s_\alpha + k);
   \]

3. **Raising the independent variable to a real power**
   \[
   \mathcal{M}_\alpha(f(t^k), s_\alpha) = \frac{1}{k}\mathcal{M}_\alpha(f(t), \frac{s_\alpha}{k});
   \]

4. **Inverse of independent variable**
   \[
   \mathcal{M}_\alpha(t^{-1}f(t^{-1}), s_\alpha) = \mathcal{M}_\alpha(f(t), 1 - s_\alpha);
   \]

5. **Multiplication by a power \( k \in \mathbb{N} \)**
   \[
   \mathcal{M}_\alpha((\ln t)^k f(t), s_\alpha) = \frac{d^k}{ds^k}\mathcal{M}_\alpha(f(t), s_\alpha);
   \]

6. **Derivative multiplied by independent variable**
   \[
   \mathcal{M}_\alpha(t^k \frac{d^k}{dt^k} f(t), s_\alpha) = (-1)^k s_\alpha^k\mathcal{M}_\alpha(f(t), s_\alpha);
   \]

7. **The Mellin transform of the convolution product**
   \[
   \mathcal{M}_\alpha(f(t) \ast g(t), s_\alpha) = \mathcal{M}_\alpha(\int_0^\infty f(\frac{t}{u})g(u)\frac{du}{u}; s_\alpha) = \mathcal{M}_\alpha(f(t), s_\alpha)\mathcal{M}_\alpha(g(t), s_\alpha).
   \]
Proof:
1. Making \( u = kt, \ du = kdt \), we have

\[
\mathcal{M}_\alpha(f(kt), s_\alpha) = \int_0^\infty f(kt)t^{s_\alpha - 1}dt = \int_0^\infty f(u)\left(\frac{u}{k}\right)^{s_\alpha - 1}du
\]

\[
= \int_0^\infty f(u)u^{s_\alpha - 1}k^{-s_\alpha}du = k^{-s_\alpha}\int_0^\infty f(u)u^{s_\alpha - 1}du = k^{-s_\alpha}\mathcal{M}_\alpha(f(t), s_\alpha).
\]

2. From direct application of Def. 2.1

\[
\mathcal{M}_\alpha(t^k f(t), s_\alpha) = \int_0^\infty f(t)t^{k_s_\alpha - 1}dt = \int_0^\infty f(t)t^{k+s_\alpha - 1}dt
\]

\[
= \int_0^\infty f(t)\left(\frac{1}{k}\right)^{s_\alpha - 1}u^{s_\alpha - 1}du = \frac{1}{k}\mathcal{M}_\alpha(f(t), s_\alpha + k).
\]

3. Considering the change of variable \( x = t^k, \ dx = kt^{k-1}dt \) and taking into account that \( 1 = x^{-\frac{1}{k}}x^{\frac{1}{k}} \)

\[
\mathcal{M}_\alpha(f(t^k), s_\alpha) = \int_0^\infty f(t^k)(t^{k}\)^{s_\alpha - 1}dt = \int_0^\infty f(x)(x^{\frac{1}{k}})^{s_\alpha - 1}\frac{1}{k}t^{1-k}dx
\]

\[
= \int_0^\infty f(x)(x^{\frac{1}{k}})^{s_\alpha - 1}\frac{1}{k}x^{\frac{1}{k} - 1}dx = \frac{1}{k}\int_0^\infty f(x)x^{\frac{s_\alpha - 1}{k}}dx = \frac{1}{k}\mathcal{M}_\alpha(f(t), s_\alpha).
\]

4. Making the change of variable \( u = t^{-1}, \ du = -t^{-2}dt \) we obtain

\[
\mathcal{M}_\alpha(t^{-1} f(t^{-1}), s_\alpha) = \int_0^\infty t^{-1} f(t^{-1})t^{s_\alpha - 1}dt = -\int_0^\infty uf(u)u^{1-s_\alpha}u^{-2}du
\]

\[
= \int_0^\infty f(u)u^{1-s_\alpha}du = \mathcal{M}_\alpha(f(t), 1-s_\alpha).
\]

5. In the particular case of \( k = 1 \), the general case can be considered by induction process,

\[
\mathcal{M}_\alpha((ln) f(t), s_\alpha) = \int_0^\infty (ln) f(t)t^{s_\alpha - 1}dt = \frac{d}{ds}\int_0^\infty f(t)t^{s_\alpha - 1}dt = \frac{d}{ds}\mathcal{M}_\alpha(f(t), s_\alpha).
\]

6. In particular, if \( k = 1 \), we have, using integration by parts on the Lizorkin space

\[
\mathcal{M}_\alpha(t\frac{d}{dt} f(t), s_\alpha) = \int_0^\infty tf'(t)t^{s_\alpha - 1}dt = \int_0^\infty f'(t)t^{s_\alpha}dt
\]

\[
- s_\alpha \int_0^\infty f(t)t^{s_\alpha - 1}dt = -s_\alpha\mathcal{M}_\alpha(f(t), s_\alpha).
\]

7. It easy to chek out the \( \alpha \)-propertie of convolution.
In [8] the definition of a ◦-product of convolution has been given for every \( f, g \in L^1(\mathbb{R}^+) \) by
\[
(f \circ g)(t) = \int_{t}^{\infty} f(x-t)g(x)dx, \quad t \geq 0.
\]
Then, considering the relation between the \( \alpha \)-Mellin Transform and the \( \alpha \)-Fourier transform obtained in [5], we have for \( f, g \in \Phi(\mathbb{R}) \)
\[
\mathcal{M}_\alpha(f \circ g)(\omega) = \mathcal{F}_\alpha( (f \circ g)(e^{-x})e^{-ax}, \omega^{\frac{1}{2}}) = \left( \int_{0}^{\infty} e^{i\omega^{\frac{1}{2}}\xi} f(\xi)d\xi \right) \mathcal{M}_\alpha(g)(\omega).
\]

3 An Application: Black-Scholes Equation

Consider us now the Black-Scholes equation [3] as the Cauchy problem
\[
\begin{cases}
  u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, & (x, t) \in (0, \infty) \times [0, T], \\
  u(x, T) = h(x), & x \in (0, \infty), \\
  u(0, t) = 0, & t \in [0, T].
\end{cases}
\tag{15}
\]
where \( \sigma, r \) and \( T \) are the constants resulting of the model, \( h \) is a suitable measurable function ; \( T \) is the time of maturity of the option. In order to solve this problem we have to obtain the price of the option. In recent several papers, the Cauchy problem, equation (15), has been considered from others point of view, for example, the authors in [1], [2], [7] and [9] have studied the same problem, considering it as an evolution equation, through semigroups of operator theory, while in [4], it has been studied using quantum mechanical supersymmetric techniques.

The differential partial equation in eq.(15) can be written as
\[
\frac{\partial}{\partial t} u(x, t) + \frac{1}{2}\sigma^2 (x \frac{\partial}{\partial x})^2 u(x, t) + (r - \frac{1}{2}\sigma^2)(x \frac{\partial}{\partial x})u(x, t) - ru(x, t) = 0,
\]
now, if the \( \alpha \)-Mellin transform is applied respect to the variable \( x \) and using its properties we have
\[
\frac{\partial}{\partial t} \mathcal{M}_\alpha(u(x), s_{\alpha}) + \frac{1}{2}\sigma^2 \mathcal{M}_\alpha((x \frac{\partial}{\partial x})^2 u(x), s_{\alpha}) + (r - \frac{1}{2}\sigma^2)\mathcal{M}_\alpha((x \frac{\partial}{\partial x})u(x), s_{\alpha}) - r\mathcal{M}_\alpha(u(x), s_{\alpha}) = 0
\]
from where, for every \( 0 < \alpha \leq 1 \)
\[
\frac{\partial}{\partial t} \mathcal{M}_\alpha(u(x, t), s_{\alpha}) + \left\{ \frac{1}{2}\sigma^2 s_{\alpha}^2 - (r - \frac{1}{2}\sigma^2)s_{\alpha} - r \right\}\mathcal{M}_\alpha(u(x, t), s_{\alpha}) = 0; \tag{16}
\]
taking into account the time of maturity of the option

\[ \mathcal{M}_\alpha(u(x, T), s_\alpha) = \mathcal{M}_\alpha(h(x), s_\alpha) \]

the equation (16) is a linear differential equation of first order. In such a case, we have

\[ \mathcal{M}_\alpha(u(x, t), s_\alpha) = \mathcal{M}_\alpha(h(x), s_\alpha) \exp \left\{ \frac{1}{2} \sigma^2 s_\alpha^2 - (r - \frac{1}{2} \sigma^2) s_\alpha - r \right\} (T - t) \]

then, applying the inverse \( \alpha \)-Mellin transform and the convolution product we obtain

\[ u(x, t) = h(x) \lor \mathcal{M}_\alpha^{-1}(\exp \left\{ \frac{1}{2} \sigma^2 (t - T) \left[ s_\alpha^2 + (1 - \frac{2}{\sigma^2}) s_\alpha - \frac{2r}{\sigma^2} \right] \right\}) \]

that have an integral representation as

\[ u(x, t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2} y^2} h(e^{y \sigma \sqrt{(T-t)-(r-\frac{1}{2} \sigma^2)(T-t)}}) dy. \]

4 Summary

In this work we have established an \( \alpha \)-Mellin transform and its inverse for every \( 0 < \alpha \leq 1 \), that can be reduced to the usual one in the case \( \alpha = 1 \). Moreover, the properties of the usual Mellin transform remain valid in the case of the fractional Mellin transform generalizing its definition. Finally, as an application we use the \( \alpha \)-Mellin transform in order to solve the Black-Scholes equation.

References


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