On Generator Cauchy Matrices
of GDRS/GTRS Codes

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Abstract

We show that every GDRS/GTRS codes have systematic generator matrices of
the form \( \left[ \frac{1}{\bar{A}} \right] / \left[ \frac{1}{A} \right] \), where A is a GC (Generalised Cauchy) matrix, \( \bar{A} \) is
GEC (Generalised Extended Cauchy) matrix, and \( \bar{\bar{A}} \) is GDC (Generalised
Doubly Extended Cauchy) matrix; and conversely every systematic generator
matrix of that form generates GDRS/GTRS codes.

Keywords: Cauchy matrix, Generalised Cauchy Matrix, Generalised Extended
Cauchy matrix, Generalised Doubly Extended Cauchy matrix, Generalised
Doubly Extended Reed-Solomon code, Generalised Triply Extended Reed-
Solomon code.
I. Introduction

An \([n,k,d]\) linear code over the finite field \(F=\text{GF}(q)\) is called MDS if \(d=n-k+1\), where \(q\) is a positive power of a prime number. MDS codes can be characterized in terms of their systematic generator matrices. If \(C\) be an \([n,k,d]\) code, whose systematic generator matrix \(G\) is given by \(G=[I|A]\), where \(I\) is the identity matrix of order \(k\), \(A\) is \(k \times (n-k)\) matrix, then code \(C\) is MDS if and only if every square submatrix of \(A\) is non-singular. There may be many systematic ways of building matrices with the property that every square submatrix is non-singular. One systematic way of doing this, is the Cauchy matrix construction. A matrix \(A=[a_{ij}]_{m \times n}\) is called a Cauchy matrix, if \(a_{ij}=1/ x_i+y_j\), where \(x_1,x_2,\ldots,x_m; y_1,y_2,\ldots,y_n\) belong to field \(F=\text{GF}(q)\). Therefore, Cauchy matrix \(A_0=[a_{ij}]_{m \times n}\) \(a_{ij}=1/x_i+y_j\), will be as:

\[
A_0 = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\
x_2+y_1 & x_2+y_2 & \cdots & x_2+y_n \\
\vdots & \vdots & \ddots & \vdots \\
x_m+y_1 & x_m+y_2 & \cdots & x_m+y_n
\end{bmatrix}
\]

(1)

Further, a matrix \(A=[a_{ij}]_{m \times n}\) is called an Extended Cauchy matrix, if \(A\) has a row(column) of 1’s, and, deleting this row(column) of 1’s changes matrix \(A\) to another matrix \(\hat{A}\), which is a Cauchy matrix. Therefore, this Extended Cauchy matrix \(A\) (having one row of 1’s) may be as:

\[
A = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\
x_2+y_1 & x_2+y_2 & \cdots & x_2+y_n \\
\vdots & \vdots & \ddots & \vdots \\
x_{m-1}+y_1 & x_{m-1}+y_2 & \cdots & x_{m-1}+y_n
\end{bmatrix}
\]

(2)
Similarly Extended Cauchy matrix $A = [a_{ij}]_{m \times n}$, having one column of 1’s can be displayed. Every square submatrix of an Extended Cauchy matrix $A$ is non-singular if and only if every square submatrix of the underlying Cauchy matrix $\hat{A}$ (obtained by deleting row (column) of 1’s from $A$) is non-singular.

II. Relation Between GC Matrices and GRS Codes

Let any vector $z$ be: $z = (z_1, z_2, \ldots, z_r)$. Let $D(z)$ denote the diagonal matrix of order $r$ with diagonal entries $D_{ii} = z_i$. Then an $m \times m$ matrix $A$ is called a Generalized Cauchy matrix (GC), if $A$ is of the form:

$$A = D(c)A_1D(d),$$

where $A_1$ is an $m \times n$ Cauchy matrix, $c = (c_1, c_2, \ldots, c_m)$, $d = (d_1, d_2, \ldots, d_n)$ are vectors of non-zero elements of field $F = GF(q)$. Therefore,

$$A = \begin{bmatrix} \frac{c_i d_j}{x_i + y_j} \end{bmatrix} \quad ;$$

where $c_i, d_j, x_i, y_j$ belong to field $F = GF(q), 1 \leq i \leq m, 1 \leq j \leq n$.

If all square submatrices of Cauchy matrix $A_1$ are non-singular, then all square submatrices of $A$ are also non-singular. Therefore, we can construct a systematic generator matrix for an $[n, k]$ MDS code by linking the identity matrix $I_k$ with a suitably defined $k \times (n-k)$ Generalised Cauchy (GC) matrix.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a vector of distinct elements of field $F = GF(q)$. Let $v = (v_1, v_2, \ldots, v_n)$ be a vector of non-zero, but not necessarily distinct elements of field $F = GF(q)$. Then code $C$ is called GRS, denoted by $GRS(n, k, \alpha, v)$, if it has a generator matrix of the form: $G = [G_1 \ G_2 \ldots \ G_n]$, where the $G_i$’s are columns of the form: $G_i = [v_{i1}, v_{i2}, \ldots, v_{ik-1}]_{k \times 1}$.

Roth and Seroussi (1985) proved that GRS code has a systematic generator matrix of the form $[I | A]$, where $A$ is a Generalised Cauchy (GC) matrix, and conversely every systematic matrix of that form generates a Generalised Reed Solomon (GRS) code.

**Theorem 1:** Let $C$ be a $GRS(n+1, k, \alpha, v)$ code defined by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}), v = (v_1, v_2, \ldots, v_n, v_{n+1})$. Then code $C$ has a systematic generator matrix of the form $[I | A]$, where $A$ is a $k \times (n+1-k)$ GC matrix such that $A_{ij} = \frac{c_i d_j}{x_i + y_j}$ with:

$$x_i = -\alpha_i, 1 \leq i \leq k$$  (5)
\[ y_j = \alpha_{j+k}, 1 \leq j \leq (n+1-k) \quad (6) \]

\[ c_j = \prod_{1 \leq i \leq k} \frac{v_{j-i}}{(\alpha_i - \alpha_j)}, \quad 1 \leq i \leq k \quad (7) \]

\[ d_j = v_{j+k} \prod_{1 \leq i \leq k} (\alpha_{j+k} - \alpha_i), \quad 1 \leq j \leq (n+1-k) \quad (8) \]

Conversely, if \( A \) is a \((n+1-k) \times k\) GC matrix defined by vectors \( x=(x_i)_{i=1}^k \), \( y=(y_j)_{j=1}^{n+1-k} \), \( c=(c_i)_{i=1}^k \), \( d=(d_j)_{j=1}^{n+1-k} \), such that every square submatrix of \( A \) is non-singular, then \([I|A]\) generates a \(\text{GRS}(n+1,k,\alpha,v)\) code with:

\[ \alpha_i = -x_i, 1 \leq i \leq k \quad (9) \]

\[ \alpha_j = y_j - k, k+1 \leq j \leq n+1 \quad (10) \]

\[ v_i = \prod_{1 \leq j \leq n+1} \frac{c_j}{(x_i - x_j)}, \quad 1 \leq i \leq k \quad (11) \]

\[ v_j = \prod_{1 \leq i \leq k} \frac{d_{j-k}}{(x_i + y_{j+k})}, \quad (k+1) \leq j \leq n+1 \quad (12) \]

**Proof:** Here \( C \) is a \(\text{GRS}(n+1,k,\alpha,v)\) code defined by: \( \alpha=(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}) \), \( v=(v_1, v_2, \ldots, v_n, v_{n+1}) \), where \( \alpha \) is a vector of distinct elements of field \( F=GF(q) \), \( v \) is a vector of non-zero (not necessarily distinct) elements of field \( F=GF(q) \). Because code \( C \) is GRS, therefore \( C \) has a generator matrix of the form: \( G=[G_1, G_2, \ldots, G_n, G_{n+1}] \), where the \( G_i \)'s are columns of the form: \( G_i=(v_1, v_i \alpha_i, v_i \alpha_i^2, \ldots, v_i \alpha_i^{k-1}) \).

Therefore, \( G=[G_1, G_2, \ldots, G_n, G_{n+1}] \)

\[ = [\alpha_j^{-1}]_{k \times (n+1)} . D(v_1, v_2, \ldots, v_n, v_{n+1}), 1 \leq j \leq n+1, 1 \leq i \leq k. \]

\[ = \bar{G}. D(v), \text{where} \; D(v)=D(v_1, v_2, \ldots, v_n, v_{n+1}) \; \text{and} \; \bar{G}=[\alpha_j^{-1}]_{k \times (n+1)}, 1 \leq j \leq n+1, 1 \leq i \leq k. \]

Now \( \bar{G}=[\alpha_j^{-1}]_{k \times (n+1)}, 1 \leq j \leq n+1, 1 \leq i \leq k \).

\[ \text{where:} \; P=[\alpha_j^{-1}]_{k \times k}, 1 \leq j \leq k, \; \text{a k} \times k \; \text{Vandermonde matrix, and} \]

\[ Q=[\alpha_j^{-1}]_{k \times (n+1-k)}, 1 \leq i \leq k, 1 \leq j \leq (n+1-k), \; \text{a k} \times (n+1-k) \; \text{matrix.} \]

Therefore, a generator matrix of code \( C=\text{GRS}(n+1,k,\alpha,v) \) is:

\[ G=\bar{G}. D(v) = [P|Q], \; D(v), \text{where} \; \bar{G}=[\alpha_j^{-1}], 1 \leq i \leq k, 1 \leq j \leq (n+1), \]

\[ P=[\alpha_j^{-1}]_{k \times k}, 1 \leq i \leq k, \; \text{a k} \times k \; \text{Vandermonde matrix.} \]

\[ Q=[\alpha_j^{-1}]_{k \times (n+1-k)}, 1 \leq i \leq k, 1 \leq j \leq (n+1-k) \]

Therefore:

\[ [I_k|A_{k \times (n+1-k)}] \sim G \sim \bar{G}. D(v) \sim [P|Q]. D(v) \quad (15) \]
Clearly $P$ is a Vandermonde matrix of order $k$. $P^{-1}$ is given by (D.E. Knuth (1969)):

$$
(P^{-1})_{ij} = \prod_{1 \leq \ell \leq i, \ell \neq j} \frac{f_{i,j-1}}{(\alpha_i - \alpha_j)}, \quad 1 \leq i, j \leq k
$$

(16)

where $f_i(z) = \prod_{1 \leq \ell \leq i, \ell \neq \ell} (z-\alpha_\ell) = \sum_{0 \leq r \leq k-1} f_{ir} z^r$ (17)

We can take Vandermonde matrices of various orders, verify that all the entries of these matrices are in accordance with (16)-(17), and hence we can verify the formulations (16)-(17). We can also verify that $A = (D(u))^{-1}.P^{-1}.Q.D(w)$, by taking any values of $(n+1)$ and $k$, and by making generalization. Therefore, systematic generator matrix of GRS(n+1,k,α,v) code $C$ is $[I \mid A]$, where:

$$
A = (D(u))^{-1}.P^{-1}.Q.D(w)
$$

(18)

So, using (5), (6), (7), (8), (14), (16), (17), we will obtain $(i,j)^{th}$ entry of $A$ as:

$$
A_{ij} = [(D(u))^{-1}.P^{-1}.Q.D(w)]_{ij} = \frac{c_i d_j}{x_i + y_j}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq (n+1-k).
$$

Therefore, $A$ is a $k \times (n+1-k)$ GC matrix.

Conversely:

Now given is that $A$ is a $k \times (n+1-k)$ GC matrix defined by vectors $x = (x_i)_{i=1}^k$, $y = (y_j)_{j=1}^{(n+1-k)}$, $c = (c_i)_{i=1}^k$, $d = (d_j)_{j=1}^{(n+1-k)}$, such that every square submatrix of $A$ is non-singular. Then reversing the steps in the first part of proof, we shall arrive at the conclusion that $[I \mid A]$ generates a GRS(n+1,k,α,v) code, where $\alpha$ and $v$ can be derived from equations (5)-(8) as follows:

Equation (5) is: $x_i - \alpha_i, (1 \leq i \leq k) \Rightarrow \alpha_i = x_i, (1 \leq i \leq k)$, which is equation (9).

Equation (6) is: $y_j - \alpha_j, (1 \leq j \leq (n+1-k)) \Rightarrow y_j = \alpha_j, (k+1 \leq j \leq n+1)$, which is equation (10).

Equation (7) is: $v_i = \prod_{1 \leq \ell \leq i, \ell \neq \ell} \frac{v_{i-1}}{(\alpha_i - \alpha_{\ell})}, \quad 1 \leq i \leq k \Rightarrow$

$$
v_j = \prod_{1 \leq \ell \leq j, \ell \neq \ell} \frac{c_i^{-1}}{(x_i - x_j)}, \quad 1 \leq i \leq k \quad \text{(using (9))} = \prod_{1 \leq \ell \leq j, \ell \neq \ell} \frac{c_i^{-1}}{(x_i - x_j)}, \quad 1 \leq i \leq k \quad \text{which is equation (11).}
$$

Equation (8) is:

$$
d_j = v_{j+k} \cdot \prod_{1 \leq \ell \leq k} (\alpha_{j+k} - \alpha_{\ell}), \quad 1 \leq j \leq (n+1-k) \Rightarrow v_{j+k} = \prod_{1 \leq \ell \leq (n+1-k)} \frac{d_j}{(\alpha_{j+k} - \alpha_{\ell})}, \quad 1 \leq j \leq n+1-k
$$
Changing \( j \) to \( j-k \), and using (9),(10), we will get:

\[
\Rightarrow v_j = \prod_{i \in \text{rank}} (x_i + y_{j-k}), 1 \leq j \leq n+1, \text{which is equation (12)}.
\]

III. Relation Between GEC Matrices and GDRS Codes

An \( m \times n \) matrix \( A \) is called a Generalised Extended Cauchy (GEC) matrix, if \( A \) is of the form:

\[
A = D(c)A_2D(d),
\]

where \( A_2 \) is an \( m \times n \) extended cauchy matrix, \( c=(c_1,c_2,\ldots,c_m), d=(d_1,d_2,\ldots,d_n) \) are vectors of non-zero elements of field \( F=GF(q) \). Therefore,

\[
A = \begin{pmatrix}
    c_1d_1 & c_1d_2 & \cdots & c_1d_n \\
    c_2d_1 & c_2d_2 & \cdots & c_2d_n \\
    x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{m-1} + y_1 & x_{m-1} + y_2 & \cdots & x_{m-1} + y_n \\
\end{pmatrix}
\]

(19)

If all square submatrices of Extended Cauchy matrix \( A_2 \) are non-singular, then all square submatrices of \( A \) are also non-singular. Therefore, we can construct a systematic generator for an \([n,k]\) MDS code by linking the identity matrix \( I_k \) with a suitably defined \( k \times (n-k) \) Generalised Extended Cauchy(GEC) matrix.

A generator matrix of Extended GRS code is a generator matrix of \( \text{GRS}(n,k,\alpha,\nu) \) code, when one of \( \alpha_i \)’s is zero. Suppose \( \alpha_n=0 \). So, a generator matrix of Extended GRS code will be:

\[
G = \begin{pmatrix}
    v_1 & v_2 & \cdots & v_n \\
    v_1\alpha_1 & v_2\alpha_2 & \cdots & 0 \\
    v_1\alpha_1^2 & v_2\alpha_2^2 & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    v_1\alpha_1^{k-1} & v_2\alpha_2^{k-1} & \cdots & 0 \\
\end{pmatrix}
\]

(20)
Code can be further extended by allowing a column of $G$ of the form: $G = (0 \, 0 \, \ldots \, 0, v_\infty)$, where $v_\infty$ is a non-zero element of field $F = \text{GF}(q)$, as a result of which MDS property is preserved. The resulting code is called Generalised Doubly Extended Reed-Solomon code, denoted by GDRS $(n+1,k,\alpha,v)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_\infty, \alpha_s, \ldots, \alpha_n)$ and $v = (v_1, v_2, \ldots, v_{s-1}, v_\infty, v_s, \ldots, v_n)$, where $s$ is the index of $G_\infty$ in $G$. Therefore, a generator matrix of GDRS $(n+1,k,\alpha,v)$ may be:

\[ G = \begin{bmatrix}
 v_1 & v_2 & \ldots & v_s & 0 \\
 v_1\alpha_1 & v_2\alpha_2 & \ldots & 0 & 0 \\
 v_1\alpha_1^2 & v_2\alpha_2^2 & \ldots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 v_1\alpha_1^{k-1} & v_2\alpha_2^{k-1} & \ldots & 0 & v_\infty^{-1}v_{k(n+1)}^{-1}
\end{bmatrix} \]  

(Roth and Serorussi (1985) proved that GDRS code has a generator systematic matrix of the form $[I|A]$, where $A$ is a Generalised Extended Cauchy(GEC) matrix, and, conversely, every systematic generator matrix of that form generates a GDRS code.

### IV. Relation Between GDC Matrices and GTRS Codes

An $m \times n$ matrix $A$ is called a Doubly Extended Cauchy matrix, if $A$ has two rows(columns) of 1’s, and deleting these two rows (columns) of 1’s changes matrix $A$ into another matrix $A'$, which is a Cauchy matrix. Therefore, a Doubly Extended Cauchy matrix, having two rows of 1’s will be as:

\[ \begin{bmatrix}
 1 & 1 & \ldots & 1 \\
 1 & 1 & \ldots & 1 \\
 \frac{1}{x_1 + y_1} & \frac{1}{x_1 + y_2} & \ldots & \frac{1}{x_1 + y_n} \\
 \frac{1}{x_2 + y_1} & \frac{1}{x_2 + y_2} & \ldots & \frac{1}{x_2 + y_n} \\
 \frac{1}{x_{m-2} + y_1} & \frac{1}{x_{m-2} + y_2} & \ldots & \frac{1}{x_{m-2} + y_n + \text{error}}
\end{bmatrix} \]  

(22)
An \( m \times n \) matrix \( A \) is called a Generalised Doubly Extended Cauchy (GDC) matrix, if it is of the form: \( A = D(c)A_3D(d) \), where \( A_3 \) is an \( m \times n \) Doubly Extended Cauchy matrix, \( c = (c_1, c_2, \ldots, c_m) \), \( d = (d_1, d_2, \ldots, d_n) \) are vectors of non-zero elements of field \( F = \text{GF}(q) \).

If all square submatrices (of order \( >2 \)) of Doubly Extended Cauchy matrix \( A_3 \) are non-singular, then all square submatrices (of order \( >2 \)) of \( A \) are also non-singular. Therefore, we can construct a systematic generator for an \([n,k]\) MDS code by linking the identity matrix \( I_k \) with a suitably defined \( k \times (n-k) \) Generalised Doubly Extended Cauchy (GDC) matrix.

The generator matrix of code GDRS(\( n+1,k,\alpha, v \)) can be further extended by allowing a more column of \( G \) of the form: \( G, G^\infty = (0 0 0 \ldots 0 v_\infty) \), where \( v_\infty \) is a non-zero element of field \( F = \text{GF}(q) \), such that MDS property is preserved. The resulting code is called Generalised Triply Extended Reed-Solomon code, denoted by GTRS (\( n+2,k,\alpha, v \)), where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1}, \alpha_\infty, \alpha_\infty^\prime, \alpha_s, \ldots, \alpha_n) \) and \( v = (v_1, v_2, \ldots, v_{s-1}, v_\infty, v_\infty^\prime, v_s, \ldots, v_n) \), where \( s \) is the index of \( G, G^\infty \) in \( G \).

**Theorem 2:** Let \( C \) be a GTRS (\( n+2,k,\alpha, v \)) code defined by \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1}, \alpha_\infty, \alpha_\infty^\prime, \alpha_s, \ldots, \alpha_n) \) and \( v = (v_1, v_2, \ldots, v_{s-1}, v_\infty, v_\infty^\prime, v_s, \ldots, v_n) \), where \( k \leq s \leq n+2 \). Then \( C \) has a generator matrix of the form \( [I | A] \), where \( A = A_\infty = (a_1, a_2, \ldots, a_k), A_\infty^\prime = (a_1, a_2, \ldots, a_k) \) before the \((s-k)\)th column of \( A \) if \( s < n+2 \), or as the last column if \( s = n+2 \), and \( a_i \)'s are as defined in (7).

**Proof:** In Theorem 1, code \( C \) was GDRS(\( n+1,k,\alpha, v \)). Here, code \( C \) is GTRS (\( n+2,k,\alpha, v \)), defined by \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1}, s, \alpha_s, \alpha_\infty, \alpha_\infty^\prime, \alpha_s, \ldots, \alpha_n) \) and \( v = (v_1, v_2, \ldots, v_{s-1}, v_\infty, v_\infty^\prime, v_s, \ldots, v_n) \), where \( k \leq s \leq n+2 \). So, generator matrix of GTRS(n+2,k,\alpha,v) code contains two additional columns: \( G^\infty = (0 0 0 \ldots 0 v_\infty) \) and \( G_\infty^\prime = (0 0 0 \ldots 0 v_\infty^\prime) \) as compared to that of GRS code. Therefore, generator matrix of GTRS (n+2,k,\alpha,v) code may be like this:

\[
G = \begin{bmatrix}
    v_1 & v_2 & \cdots & v_{s-1} & 0 & 0 & v_s & \cdots & v_n \\
    v_1 \alpha_1 & v_2 \alpha_2 & \cdots & v_{s-1} \alpha_{s-1} & 0 & 0 & v_s \alpha_s & \cdots & v_n \alpha_n \\
    \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    v_1 \alpha_1^{k-1} & v_2 \alpha_2^{k-1} & \cdots & v_{s-1} \alpha_{s-1}^{k-1} & v_\infty & v_\infty^\prime & v_s \alpha_s^{k-1} & \cdots & v_n \alpha_n^{k-1}
\end{bmatrix}
\]

Because \( k \leq s \leq n+2 \), and \( k \) is the number of message-symbols, so \( G_\infty \) and \( G_\infty^\prime \) will appear among the columns of \( G \) corresponding to check-digits.
Therefore, \( G_\infty \) and \( G_\infty / \) will correspond respectively to columns \( A_\infty \) and \( A_\infty / \) given by:

\[
A_\infty = D(u)^{-1} \cdot P^{-1} \cdot G_\infty ; \quad A_\infty / = D(u)^{-1} \cdot P^{-1} \cdot G_\infty /
\]

in the systematic generator matrix of code \( C \) i.e. in \( [I_{k \times k} | A_{k \times (n-k+2)}] \).

So, \( A_\infty = D(u)^{-1} \cdot (P^{-1})_k \cdot v_\infty \) and \( A_\infty / = D(u)^{-1} \cdot (P^{-1})_k \cdot v_\infty / \),

where \( (P^{-1})_k \) denotes the \( k \)th column of \( P^{-1} \), and \( u = (v_1, v_2, \ldots, v_k) \).

Therefore,

\[
A_\infty = v_1^{-1} \cdot (P^{-1})_{ik} \cdot v_\infty \quad \text{(24)}
\]

and

\[
A_\infty / = v_\infty^{-1} \cdot (P^{-1})_{ik} \cdot v_\infty / \quad \text{(25)}
\]

Consider the polynomial:

\[
f_i(z) = \prod_{1 \leq k \neq l \leq k} (z - \alpha_i) = \sum_{0 \leq r \leq k-1} f_{i,r} z^r \quad \text{(26)}
\]

\( P^{-1} \) is given by [D.E. Knuth (1969)]:

\[
(P^{-1})_{ij} = \prod_{1 \leq k \neq l \leq i} \frac{f_{i,k-1}}{(\alpha_i - \alpha_l)}, \quad 1 \leq i, j \leq k \quad \text{(27)}
\]

Therefore (24) implies:

\[
A_{ik} = v_1^{-1} \cdot \prod_{1 \leq k \neq l \leq i} \frac{f_{i,k-1}}{(\alpha_i - \alpha_l)} \cdot v_\infty, \quad 1 \leq i \leq k \quad \text{(28)}
\]

Using (26) and (7) in (28), we get:

\[
A_{ik} = v^{-1}_i \cdot \prod_{1 \leq k \neq l \leq i} \frac{v_1^{-1}}{(\alpha_i - \alpha_l)} \cdot v_\infty, \quad 1 \leq i \leq k
\]

\[
= (d_\infty) \cdot (c_1, c_2, \ldots, c_k)^\prime.
\]

Therefore, \( A_\infty = (d_\infty) \cdot (c_1, c_2, \ldots, c_k)^\prime \).

Similarly (25) implies (using (7): 

\[
A_{ik} / = v_\infty^{-1} \cdot (P^{-1})_{ik} \cdot v_\infty / = (d_\infty) \cdot (c_1, c_2, \ldots, c_k)^\prime.
\]

Therefore, \( A_\infty / = (d_\infty) \cdot (c_1, c_2, \ldots, c_k)^\prime \).


**References**


**Received: September, 2012**