The Generalized Pascal-Like Triangle 
and Applications

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Abstract
We construct the generalized Pascal-like triangle and derive the explicit formulas for the second order linear recurrences by using some properties of this triangle. Applications to earlier results about generalized Fibonacci and Lucas numbers.

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Keywords: Fibonacci numbers, Lucas numbers

1 Introduction
The second order linear recurrence sequence \( \{W_n\}_{n \geq 0} \) of real numbers is defined by

\[
W_{n+2} = aW_{n+1} + bW_n
\]

where \( W_0 = p \) and \( W_1 = q \).

If \( p = 0, q = 1 \), then \( W_n = U_n \) is the generalized Fibonacci numbers. If \( p = 2, q = a \), then \( W_n = V_n \) is the generalized Lucas numbers. For \( a = b = 1 \), \( U_n \) and \( V_n \) are the well-known Fibonacci numbers \( F_n \) and Lucas numbers \( L_n \), respectively.

It is well-known that the explicit formulas for the generalized Fibonacci and Lucas numbers are

\[
U_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i} a^{n-2i} b^i, \quad V_n = \sum_{i=0}^{[n/2]} \frac{n}{n-i} \binom{n-i}{i} a^{n-2i} b^i,
\]

respectively, see the equations (2.7) and (2.8) in [2], also [1].

In this paper we consider the second order linear recurrent sequence and derive the explicit formula for this sequence by using the Pascal-like triangle which is defined.
2 The generalized Pascal-like triangle

Definition 2.1. Let $n$ be a positive integer. For $i \in \mathbb{Z}$, the $A_{n,i}$ is defined as

$$A_{n,i} = \begin{cases} a^{n-1}q & ; i = 0 \\ b^n p & ; n = i \\ aA_{n-1,k} + bA_{n-1,k-1} & ; 0 < i < n \\ 0 & ; \text{otherwise} \end{cases}$$

(2)

We see that $aA_{n,0} = A_{n+1,0}$ and $bA_{n,n} = A_{n+1,n+1}$.

Definition 2.2. The generalized Pascal-like triangle is defined as follows

|   | 0   | 1      | 2      | 3      | 4      | ... | n      | ...
<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_{1,0}$</td>
<td>$A_{1,1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$A_{2,0}$</td>
<td>$A_{2,1}$</td>
<td>$A_{2,2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$A_{3,0}$</td>
<td>$A_{3,1}$</td>
<td>$A_{3,2}$</td>
<td>$A_{3,3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$A_{4,0}$</td>
<td>$A_{4,1}$</td>
<td>$A_{4,2}$</td>
<td>$A_{4,3}$</td>
<td>$A_{4,4}$</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>$A_{n,0}$</td>
<td>$A_{n,1}$</td>
<td>$A_{n,2}$</td>
<td>...</td>
<td>$A_{n,n}$</td>
<td></td>
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</tr>
</tbody>
</table>

The following triangle will be shown any elements of the Pascal-like triangle in the variables $a, b, p$ and $q$ by using Definition 2.1.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q$</td>
<td>$bp$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$aq$</td>
<td>$abp + bq$</td>
<td>$b^2p$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$a^2q$</td>
<td>$a^2bp + 2abq$</td>
<td>$2a^2bp + b^2q$</td>
<td>$b^3p$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$a^3q$</td>
<td>$a^3bp + 3a^2bq$</td>
<td>$3a^2b^2p + 3ab^2q$</td>
<td>$3ab^3p + b^2q$</td>
<td>$b^4p$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$a^4q$</td>
<td>$a^4bp + 4a^3bq$</td>
<td>$4a^3b^2p + 6a^2b^2q$</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$a^5q$</td>
<td>$a^5bp + 5a^4bq$</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$a^6q$</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>...</td>
<td></td>
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</tbody>
</table>

We see that the sums of elements in each row of the Pascal-like triangle is $(a + b)^{n-1}(q + bp)$.

The following theorem gives an alternative definition of $A_{n,i}$ as the binomial sums.

Theorem 2.3. Let $n \in \mathbb{N}$ and $0 \leq i \leq n$. Then

$$A_{n,i} = a^{n-i}b^i p \binom{n-1}{i-1} + a^{n-i-1}b^i q \binom{n-1}{i}.$$  (3)
Proof. For \( n = 1 \), we see that the equation (3) holds for \( i = 0, 1 \). By induction on \( n \), assume that (3) is true for \( k \in \mathbb{N} \) and \( 0 \leq i \leq k \). By (2) and the inductive hypothesis, we get

\[
A_{k+1,i} = aA_{k,i} + bA_{k,i-1}
\]

\[
= a \left[ a^{k-i}i^p \binom{k-1}{i-1} + a^{k-i-1}i^q \binom{k-1}{i} \right] + b \left[ a^{k-i+1}i^p \binom{k-1}{i-2} + a^{k-i-1}i^q \binom{k-1}{i-1} \right]
\]

\[
= a^{k-i+1}i^p \left[ \binom{k-1}{i-1} + \binom{k-1}{i-2} \right] + a^{k-i}i^q \left[ \binom{k-1}{i} + \binom{k-1}{i-1} \right]
\]

\[
= a^{k-i+1}i^p \binom{k}{i-1} + a^{k-i}i^q \binom{k}{i},
\]

showing that (3) works for \( n = k + 1 \).

\[\square\]

3 Explicit formulas

In this section we derive the explicit formulas for the second order recurrence sequence.

**Theorem 3.1.** Let \( n \in \mathbb{N} \). We have

\[
W_n = \sum_{i=0}^{\lfloor n/2 \rfloor} A_{n-i,i}.
\]

(4)

Proof. For \( n = 1 \), we see that the equation (4) holds. By induction on \( n \), assume that (4) is true for \( k \in \mathbb{N} \) and \( 0 \leq i \leq k \). By (1),(2) and the inductive hypothesis, we get

\[
W_{k+1} = aW_k + bW_{k-1}
\]

\[
= a \sum_{i=0}^{\lfloor k/2 \rfloor} A_{k-i,i} + b \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} A_{k-i-1,i}
\]

\[
= aA_{k,0} + a \sum_{i=1}^{\lfloor k/2 \rfloor} A_{k-i,i} + b \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} A_{k-i,i-1}
\]

\[
= \begin{cases} 
A_{k+1,0} + \sum_{i=1}^{\lfloor k/2 \rfloor} A_{k-i+1,i} & \text{; } k \text{ is even} \\
A_{k+1,0} + \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} A_{k-i+1,i} + bA_{\frac{k-1}{2},\frac{k-1}{2}} & \text{; } k \text{ is odd}
\end{cases}
\]
By using Theorem 2.3, we can write $W_n$ in the binomial sum.

**Corollary 3.2.** Let $n \in \mathbb{N}$. We have

$$W_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[ a^{n-2i} b^i \binom{n-i-1}{i-1} + a^{n-2i-1} b^i \binom{n-i-1}{i} \right].$$

Particular cases for the Corollary 3.2:

- If we take $p = 0$ and $q = 1$, then $W_n = U_n$ and

$$U_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} a^{n-2i-1} b^i.$$

- If we take $p = 2$ and $q = a$, then $W_n = V_n$ and

$$V_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[ 2a^{n-2i} b^i \binom{n-i-1}{i-1} + a^{n-2i-1} b^i \binom{n-i-1}{i} \right]$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[ \binom{n-i}{i} + \binom{n-i-1}{i-1} \right] a^{n-2i} b^i$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[ \binom{n-i}{i} + \frac{i}{n-i} \binom{n-i}{i} \right] a^{n-2i} b^i$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} a^{n-2i} b^i.$$

Two above identities are two identities in section 1.

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**References**


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