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A Fundamental Theorem of Set-valued Homomorphism of Groups

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Abstract

In this paper, we study some properties of lower and upper T- rough normal sub groups based on a set valued homomorphism T from a group G_1 to the set of all non-empty subsets of a group G_2. We also, prove the isomorphism theorem for lower and upper T-rough subgroups of G_1 and G_2. We introduce the concept of the kernel of the set-valued homomorphism and proved that it is a normal subgroup of G_1. As a main result of this paper, we prove the analog of the fundamental theorem of homomorphism of groups to the set valued homomorphism of groups.

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1 Introduction

Z. Pawlak [20], in his pioneering paper, introduced the theory of rough sets as a tool to model uncertainty, vague and incomplete information system. Since then, the theory of rough sets attracted many researchers in mathematics, computer science and engineering. Originally rough sets are described as lower and upper approximations based on equivalence relations. Later, generalized rough sets were considered based on arbitrary relations by W. Zhu [27].

Algebraic systems on rough sets were considered by mathematicians like R. Biswas, S. Nanda, N. Kuroki, P. P. Wang, B. Davvaz and many others. R. Biswas and S. Nanda [1] introduced the concept of rough groups and rough subgroups. N. Kuroki [15] introduced the notion of rough ideal in a semigroup. N. Kuroki and P. P. Wang [16] studied the lower and upper approximations in a fuzzy group. Rough sets and rough groups were also considered by N. Kuroki and J. N. Moderson [17]. B. Davvaz have introduced roughness in many algebraic systems. Considering ring as a universal set, B. Davvaz [3] introduced the notion of rough ideals and rough sub rings with respect to an ideal of the ring. In [12], O. Kazanci and B. Davvaz introduced the notion of rough prime ideals and rough primary ideals in a ring. The authors V. Selvan and G. Senthil Kumar [23, 24] have studied the roughness of ideals and fuzzy ideals in a semiring. In [7], B. Davvaz introduced the concept of set valued homomorphism and T-rough sets in a group. In [26], S. Yamak, O. Kazanci and B. Davvaz introduced the generalized lower and upper approximations in a ring based on set valued homomorphism of rings. S. B. Hosseini et al [8,9] have studied some properties of T- rough sets in semigroups, commutative rings.

In this paper, we consider the set-valued homomorphism on groups and study some properties of it. In [7], B. Davvaz proved that if $T : G_1 \to P^*(G_2)$ is a set-valued homomorphism from a group G_1 to the set of all non-empty subsets of a group G_2 and if H is a normal subgroup of G_2, then the upper approximation of H, viz, $U_T(H)$ is a normal subgroup of G_1. In section 3 of this paper, assuming H a normal subgroup of G_2 containing $T(e_1)$, where e_1 is the identity element of G_1, we prove that the lower approximation of H, viz, $L_T(H)$ is also a normal subgroup of G_1. Also, if $f : G_1 \to G_2$ is a homomorphism from a group G_1 onto a group G_2 and if $T_2 : G_2 \to P^*(G_2)$ is a set-valued homomorphism then $T_1 : G_1 \to P^*(G_1)$ defined by $T_1(x) = \left\{ g_1 \in G_1 \middle| f(g_1) \in T_2(f(x)) \right\}, \forall x \in G_1$ is a set-valued homomorphism and if H is a normal subgroup of G_1 containing $T(e_1)$ we further prove that $G_1/L_{T_1}(H) \cong G_2/L_{T_2}(f(H))$ and $G_1/U_{T_1}(H) \cong G_2/U_{T_2}(f(H))$. In section 4, we introduce the concept of kernel of a set-valued homomorphism $T : G_1 \to P^*(G_2)$ and prove that it is a normal subgroup of G_1. As a main result of this paper, we prove the analog of the fundamental theorem of homomorphism of groups for the set-valued homomorphism.

2 Preliminaries

The set-valued mappings, the lower and upper approximation with respect to a set-valued mapping and the set-valued homomorphism of groups are introduced by B. Davvaz and can be found in [7]. We recall them for the sake of completeness.

Definition 2.1 [7] Let X and Y be two non-empty sets and $B \subseteq Y$. Let $T : X \to P^*(Y)$ be a set-valued mapping, where $P^*(Y)$ denotes the set of all non-empty subsets of Y. The lower and upper approximations of B under T is given by $L_T(B) = \{x \in X \mid T(x) \subseteq B\}$; $U_T(B) = \{x \in X \mid T(x) \cap B \neq \varnothing\}$

Definition 2.2 [7] Let X and Y be two non-empty sets and $B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping, then the pair$(L_T(B), U_T(B))$ is referred to as the generalized rough set of B induced by T.

Definition 2.3 [7] A set-valued homomorphism T from a group G_1 to a group G_2 is a mapping from G_1 into $P^*(G_2)$ that preserves the group operation, that is, $T(ab) = T(a)T(b)$ and $(T(a))^{-1} = \{x^{-1} \mid x \in T(a)\} = T(a^{-1})$, for all $a, b \in G_1$.

Example 2.4 Every homomorphism of groups can be considered as a set valued homomorphism of groups. For, if $f : G_1 \to G_2$ is a homomorphism of groups then $T_f : G_1 \to P^*(G_2)$ defined by $T_f(x) = \{f(x)\}$, $\forall\, x \in G_1$ is a set-valued homomorphism from G_1 to $P^*(G_2)$.

For more examples of set-valued homomorphism of groups, one can refer [7].

Definition 2.5 [7] Let G be a group. Let θ be a congruence of G, that is, θ is an equivalence relation on G such that $(a, b) \in \theta$ implies $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$ for all $x \in G$. We denote by $[a]_\theta$ the θ-congruence class containing the element $a \in G$.

Example 2.6 Let G be a group, H a normal subgroup of G. For $a, b \in G$, we define $a \equiv b \pmod H$ if and only if $ab^{-1} \in H$. Then the relation $\equiv$ is congruence relation on G.

Proposition 2.7 [7] Let θ be a congruence on a group G. If $a, b \in G$ then

(i) $[a]_\theta [b]_\theta = [ab]_\theta$ and (ii) $[a^{-1}]_\theta = \{[a]_\theta\}^{-1}$.

Corollary 2.8 [7] Let θ be a congruence on a group G. Define $T : G \to P^*(G)$ by $T(x) = [x]_\theta$, $\forall\, x \in G$. Then T is a set-valued homomorphism.

Definition 2.9 [7] Let T be a set-valued mapping from G_1 into $P^*(G_2)$. The mapping T is said to be lower semi-uniform if for each subgroup B in G_2 , the set $L_T(B)$ is a subgroup of G_1. The mapping T is said to be upper semi-uniform if for each subgroup B in G_2, the set $U_T(B)$ is a subgroup of G_1. A set-valued mapping T is said to be uniform if it is upper and lower semi-uniform.

Theorem 2.10 [7] Every set-valued homomorphism is uniform.

Definition 2.11 Let G_1, G_2 be two groups, $T: G_1 \to P^*(G_2)$ be a set-valued homomorphism and H be a normal subgroup of G_2. If $L_T(H)$ is a normal subgroup of G_1, then H is called the lower T-rough normal subgroup of G_2 and if $U_T(H)$ is a normal subgroup of G_1 then H is called the upper T-rough normal subgroup of G_2. If $L_T(H)$ and $U_T(H)$ are normal subgroup of G_1, then we call $(L_T(H), U_T(H))$ a T-rough normal subgroup.

3 Isomorphism theorem for T-rough Groups

Let G_1, G_2 be two groups, H be a normal subgroups of G_2 and $T: G_1 \to P^*(G_2)$ be a set-valued homomorphism. In [7, Theorem 3.11] B.Davvaz proved that $U_T(H)$ is a normal subgroup of G_1. In the following theorem, assuming H a normal subgroup of G_2 containing $T(e_1)$, where e_1 is the identity element of G_1, we prove that $L_T(H)$ is also a normal subgroup of G_1.

Theorem 3.1 Let G_1, G_2 be two groups, H be a normal subgroup of G_2 containing $T(e_1)$ and $T: G_1 \to P^*(G_2)$ be a set-valued homomorphism. Then $L_T(H)$ is a normal subgroup of G_1.

Proof. By [7] Theorem [4.10], if H is a subgroup of G_2 then $L_T(H)$ is a subgroup of G_1. Suppose $g_1 \in G_1$ and $x \in L_T(H)$ then $T(x) \subseteq H$.

$$\begin{aligned} T\left(g_1^{-1} x g_1\right) &= T\left(g_1^{-1}\right) T(x)\, T(g_1) \subseteq T\left(g_1^{-1}\right) H\ T(g_1) \\ &= T\left(g_1^{-1}\right) T(g_1) H, \quad \text{since } H \text{ is normal in } G_2 \\ &= T\left(g_1^{-1} g_1\right) H = T(e_1) H = H . \end{aligned}$$

That is, $g_1^{-1} x g_1 \in L_T(H)$, which proves that $L_T(H)$ is a normal subgroup of G_1.

Corollary 3.2 Let G_1 and G_2 be two groups, $T:G_1\to P^*(G_2)$ be a set-valued homomorphism and H a normal subgroup of G_2 containing $T(e_1)$, then H is a T-rough normal subgroup.

Theorem 3.3 Let G_1 and G_2 be two groups, $f:G_1\to G_2$ be an epimorphism and $T_2:G_2\to P^*(G_2)$ be a set-valued homomorphism. If f is one-to-one and $T_1(x)=\{g_1\in G_1 \mid f(g_1)\in T_2(f(x))\}, \forall x\in G_1$ then $T_1: G_1\to P^*(G_1)$ is a set-valued homomorphism.

Proof. Let $u\in T_1(xy)$. Then $f(u)\in T_2(f(xy)) = T_2(f(x)f(y)) = T_2(f(x))\cdot T_2(f(y))$. That is $f(u)=ab$, for some $a\in T_2(f(x))$, $b\in T_2(f(y))$. Since f is onto, there exists c, $d \in G_1$ such that $f(c)=a$, $f(d)=b$. Hence, $f(u)=f(c)f(d)$, $c\in T_1(x)$ and $d\in T_1(y)$. Therefore, $u=cd$, which implies that $u\in T_1(x)T_1(y)$. Hence, $T_1(xy)\subseteq T_1(x)T_1(y)$. Conversely, assume that $z\in T_1(x)T_1(y)$. Then $z=cd$ for some $c\in T_1(x)$ and $d\in T_1(y)$ and so, $f(c)\in T_2(f(x))$, $f(d)\in T_2(f(y))$. Therefore, $f(cd)=f(c)f(d) \in T_2(f(x))T_2(f(y)) = T_2(f(x)f(y)) = T_2(f(xy))$ which implies that, $z=cd\in T_1(xy)$. Therefore, $T_1(x)T_1(y)\subseteq T_1(xy)$. Hence, $T_1(xy)=T_1(x)T_1(y)$.

We have, $c\in T_1(x^{-1}) \Leftrightarrow f(c)\in T_2(f(x^{-1})) \Leftrightarrow f(c)\in T_2((f(x))^{-1})$

$\Leftrightarrow f(c)\in (T_2(f(x)))^{-1} \Leftrightarrow (f(c))^{-1}\in T_2(f(x)) \Leftrightarrow f(c^{-1})\in T_2(f(x))$

$\Leftrightarrow c^{-1}\in T_1(x) \Leftrightarrow c\in (T_1(x))^{-1}$. Therefore, $T_1(x^{-1})=(T_1(x))^{-1}, \forall x\in G_1$. Thus, T_1 is a set-valued homomorphism.

Theorem 3.4 Let G_1 and G_2 be two groups, $f:G_1\to G_2$ be an onto homomorphism and let $T_2:G_2\to P^*(G_2)$ be a set-valued homomorphism. If $T_1(x)=\{g_1\in G_1 \mid f(g_1)\in T_2(f(x))\}, \forall x\in G_1$ and A is a non-empty subset of G_1, then

(i) $f(U_{T_1}(A))=U_{T_2}(f(A))$

(ii) $f(L_{T_1}(A))\subseteq L_{T_2}(f(A))$. Moreover, if f is one-to-one then equality holds.

Proof. (i) Let $y\in f(U_{T_1}(A))$. Then there exists $x\in U_{T_1}(A)$ such that $y=f(x)$. As $x\in U_{T_1}(A)$, $T_1(x)\cap A\neq\varnothing$. Hence, there exists $a\in A$ with

$a \in T_1(x)$. Thus $f(a) \in T_2(f(x))$ and $f(a) \in f(A)$, which implies that $T_2(f(x)) \cap f(A) \neq \emptyset$. Therefore, $y = f(x) \in U_{T_2}(f(A))$. Conversely, if $y \in U_{T_2}(f(A))$, then since f is onto, there exists $x \in G_1$ such that $y = f(x) \in U_{T_2}(f(A))$. That is, $T_2(f(x)) \cap f(A) \neq \emptyset$. Hence there exists $z \in T_2(f(x)) \cap f(A)$. Thus, $z = f(a)$ for some $a \in A$. That is, $z = f(a) \in T_2(f(x))$. Hence, $a \in T_1(x) \cap A$, so that $T_1(x) \cap A \neq \emptyset$. This proves that $x \in U_{T_1}(A)$ and $y = f(x) \in f(U_{T_1}(A))$. Hence, $U_{T_2}(f(A)) \subseteq f(U_{T_1}(A))$. Thus $f(U_{T_1}(A)) = U_{T_2}(f(A))$.

(ii) Let $y \in f(L_{T_1}(A))$. Then there exists $x \in L_{T_1}(A)$ such that $y = f(x)$. Now, let $w \in T_2(f(x))$. Since f is onto, there exists $z \in G_1$ such that $f(z) = w$. Thus, $w = f(z) \in T_2(f(x))$, which implies that $z \in T_1(x)$. As $x \in L_{T_1}(A)$, it follows that $z \in T_1(x) \subseteq A$. Thus, $w = f(z) \in f(A)$. Hence, $T_2(f(x)) \subseteq f(A)$ which proves that $y = f(x) \in L_{T_2}(f(A))$. Conversely, let $y \in L_{T_2}(f(A))$. Then $T_2(y) \subseteq f(A)$. Since f is onto, there exists $x \in G_1$, such that $y = f(x)$. Therefore, $T_2(f(x)) \subseteq f(A)$. Now, let $u \in T_1(x)$. Then $f(u) \in T_2(f(x)) \subseteq f(A)$. Thus, there exists $a \in A$ such that $f(u) = f(a)$. Since f is an one-to-one, $u = a \in A$. Therefore, $T_1(x) \subseteq A$. That is, $x \in L_{T_1}(A)$. Hence $y = f(x) \in f(L_{T_1}(A))$, which proves that $L_{T_2}(f(A)) \subseteq f(L_{T_1}(A))$. Thus, $f(L_{T_1}(A)) = L_{T_2}(f(A))$.

Theorem 3.5 Let G_1 and G_2 be two groups, f be an isomorphism from G_1 onto G_2 and $T_2 : G_2 \to P^*(G_2)$ be a set-valued homomorphism. Let H be a subgroup of G_1. If $T_1(x) = \{g_1 \in G_1 \mid f(g_1) \in T_2(f(x))\}, \forall x \in G_1$, then

(i) $U_{T_1}(H)$ is a subgroup of G_1 if and only if $U_{T_2}(f(H))$ is a subgroup of G_2.

(ii) $U_{T_1}(H)$ is a normal subgroup of G_1 if and only if $U_{T_2}(f(H))$ is a normal subgroup of G_2.

(iii) $L_{T_1}(H)$ is a subgroup of G_1 if and only if $L_{T_2}(f(H))$ is a subgroup of G_2.

(iv) $L_{T_1}(H)$ is a normal subgroup of G_1 if and only if $L_{T_2}(f(H))$ is a normal subgroup of G_2.

Proof. Since $f: G_1 \to G_2$ is an isomorphism, a subset A is a subgroup of G_1 iff $f(A)$ is a subgroup of G_2. Hence the proof follows from Theorem 3.4.

Theorem 3.6 Let G_1 and G_2 be two groups, f be an isomorphism from G_1 onto G_2 and let $T_2 : G_2 \to P^*(G_2)$ be a set-valued homomorphism. If $T_1(x) = \{ g_1 \in G_1 \mid f(g_1) \in T_2(f(x)) \}, \forall x \in G_1$ and A a normal subgroup of G_1 containing $T_1(e_1)$ then

(i) $\dfrac{G_1}{L_{T_1}(A)} \cong \dfrac{G_2}{L_{T_2}(f(A))}$ and (ii) $\dfrac{G_1}{U_{T_1}(A)} \cong \dfrac{G_2}{U_{T_2}(f(A))}$

Proof. Define $\phi : G_1 \to \dfrac{G_2}{L_{T_2}(f(A))}$ by $\phi(g_1) = L_{T_2}(f(A))\, f(g)$, where $g_1 \in G_1$

Clearly, ϕ is well defined.

Let $g_1, g_1' \in G_1$. Then $\phi(g_1 g_1') = L_{T_2}(f(A))\, f(g_1 g_1')$

$= L_{T_2}(f(A))\, f(g_1)\, f(g_1')$, since f is a homomorphism

$= L_{T_2}(f(A))\, f(g_1)\, L_{T_2}(f(A))\, f(g_1')$, since $L_{T_2}(f(A))$ is normal in G_2

$= \phi(g_1)\phi(g_1')$

Therefore, ϕ is homomorphism.

$$\begin{aligned}
\text{Ker } \phi &= \{ g_1 \in G_1 \mid \phi(g_1) = L_{T_2}(f(A)) \} \\
&= \{ g \in G_1 \mid L_{T_2}(f(A))\, f(g_1) = L_{T_2}(f(A)) \} \\
&= \{ g \in G_1 \mid f(g_1) \in L_{T_2}(f(A)) \} = \{ g_1 \in G_1 \mid \mathrm{g}_1 \in L_{T_1}(A) \} = L_{T_1}(A)
\end{aligned}$$

Let $L_{T_2}(f(A)) \cdot g_2 \in \dfrac{G_2}{L_{T_2} f(A)}$. Since f is onto, there exists $g_1 \in G_1$ such that $f(g_1) = g_2$. Hence, $L_{T_2}(f(A))g_2 = L_{T_2}(f(A)\, f(g_1)) = \phi(g_1)$. Therefore, ϕ is onto.

By fundamental theorem of homomorphism of groups,

$$\frac{G_1}{\ker \phi} \cong \frac{G_2}{L_{T_2}(f(A))}. \quad \text{That is, } \frac{G_1}{L_{T_1}(A)} \cong \frac{G_2}{L_{T_2}(f(A))}.$$

The proof of (ii) is similar to the proof of (i).

4 Isomorphism theorem for set-valued homomorphism of Groups

In this section, we introduce the concept of the kernel of set-valued homomorphism of groups and establish the isomorphism theorems of groups in the context of set-valued homomorphism of groups.

Let G_1 and G_2 be two groups and $T : G_1 \to P^*(G_2)$ be a set-valued homomorphism. Define $K = \{ x \in G_1 \mid T(x) = T(e_1) \}$, where e_1 is the identity element of G_1. K is called the kernel of T.

Theorem 4.1 K is a normal subgroup of G_1.

Proof. Let $x, y \in K$. Then $T(x) = T(e_1)$ and $T(y) = T(e_1)$. Since T is a set-valued homomorphism, $T(xy) = T(x)T(y) = T(e_1)T(e_1) = T(e_1)$. Therefore, $xy \in K$. Also, $T(x^{-1}) = [T(x)]^{-1} = [T(e_1)]^{-1} = T(e_1^{-1}) = T(e_1)$, which proves that $x^{-1} \in K$. Now, if $g_1 \in G_1$ and $x \in K$ then $T(g_1 x g_1^{-1})$ $= T(g_1)T(x)T(g_1^{-1})$ $= T(g_1)T(e_1)T(g_1^{-1})$ $= T(g_1 e_1 g_1^{-1}) = T(e_1)$. That is, $g_1 x g_1^{-1} \in K$. Hence, K is a normal subgroup of G_1.

Theorem 4.2 Let G_1 and G_2 be two groups. If $T : G_1 \to P^*(G_2)$ is a set-valued homomorphism then $T(G_1) = \bigcup_{g_1 \in G_1} T(g_1)$ is a subgroup of G_2

Proof. Let $y_1, y_2 \in \bigcup_{g_1 \in G_1} T(g_1)$. Then there exists $g_1, g_1' \in G_1$ such that $y_1 \in T(g_1)$ and $y_2 \in T(g_1')$. Then $y_1 y_2 \in T(g_1)T(g_1')$ $= T(g_1 g_1')$ $\subseteq \bigcup_{g_1 \in G_1} T(g_1)$. Therefore, $y_1 y_2 \in T(G_1)$ and $y_1^{-1} \in T(g_1)^{-1} = T(g_1^{-1})$ $\subseteq \bigcup_{g_1 \in G_1} T(g_1)$. Hence, $T(G_1)$ is a subgroup of G_2.

Lemma 4.3 Let G_1 and G_2 be two groups and $T : G_1 \to P^*(G_2)$ a set-valued homomorphism, then $e_2 \in T(e_1)$, where e_1, e_2 are the identity elements of G_1 and G_2 respectively.

Proof. Let $g_1 \in G_1$, and let $T(g_1) = \{x_1, x_2, \ldots, x_k\} \subseteq G_2$, then $[T(g_1)]^{-1}$ $= \{x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}\} \subseteq G_2$. We have $e_2 = x_i x_i^{-1} \in T(g_1)T(g_1^{-1}) = T(g_1 g_1^{-1})$ $= T(e_1)$. That is, $e_2 \in T(e_1)$.

Lemma 4.4 Let G_1 and G_2 be two groups and $T:G_1 \to P^*(G_2)$ a set-valued homomorphism. For $A \subseteq G_1$, define $T(A) = \underset{a \in A}{\cup} T(a)$. If A is a subgroup of G_1, then $T(A)$ is a subgroup of G_2.

Proof. Let $x_1, x_2 \in G_2$ such that $x_1, x_2 \in T(A)$. Then $x_1 \in T(a_1)$ and $x_2 \in T(a_2)$, for some $a_1, a_2 \in A$. Thus, $x_1 x_2 \in T(a_1)T(a_2) = T(a_1 a_2) \subseteq T(A)$, and $x_1^{-1} \in \left[T(a_1)\right]^{-1} = T\left(a_1^{-1}\right) \subseteq T(A)$. Hence, $T(A)$ is a subgroup of G_2.

Corollary 4.5 $T(e_1)$ is a subgroup of G_2.

If $T:G_1 \to P^*(G_2)$ is a set valued homomorphism then define $T^*(G_1) = \{T(g_1) \mid \mathrm{g}_1 \in G_1\}$. It can be easily seen that $T^*(G_1)$ is a group under the operation given by $T(g_1) \circ T(g_1') = T(g_1 g_1')$.

Theorem 4.6 Let G_1 and G_2 be two groups and $T:G_1 \to P^*(G_2)$ be a set-valued homomorphism with kernel K. Then $\frac{G_1}{K} \cong T^*(G_1)$.

Proof. Define a map $\psi : \frac{G_1}{K} \to T^*(G_1)$ by $\psi(g_1 K) = T(g_1)$, where $g_1 \in G_1$.

Suppose $g_1 K = g_1' K$. Then $g_1 g_1'^{-1} \in K \Rightarrow T\left(g_1 g_1'^{-1}\right) = T(e_1)$

$\Rightarrow T(g_1) T\left(g_1'^{-1}\right) = T(e_1) \Rightarrow T(g_1)\left(T\left(g_1'\right)\right)^{-1} = T(e_1) \Rightarrow T(g_1) = T\left(g_1'\right)$

$\Rightarrow \psi(g_1 K) = \psi\left(g_1' K\right)$

Hence, ψ is well defined.

Clearly ψ is onto, since the pre image of $T(g_1) \in T^*(G_1)$ is $g_1 K$. That is, $\psi(g_1 K) = T(g_1)$. Next, let $g_1, g_1' \in G_1$, then $\psi\left(g_1 K \cdot g_1' K\right) = \psi\left(g_1 g_1' K\right)$ $= T\left(g_1 g_1'\right) = \psi(g_1 K)\psi\left(g_1' K\right)$. Therefore, ψ is homomorphism.

Now, suppose $\psi(g_1 K) = \psi\left(g_1' K\right) \Rightarrow T(g_1) = T\left(g_1'\right) \Rightarrow T(g_1) T\left(g_1'^{-1}\right)$ $= T(e_1) \Rightarrow T\left(g_1 g_1'^{-1}\right) = T(e_1) \Rightarrow g_1 g_1'^{-1} \in K \Rightarrow g_1 K = g_1^1 K$.

Therefore, ψ is one-to-one. Thus, we have proved that $\frac{G_1}{K} \cong T^*(G_1)$.

Theorem 4.7 Let $T:G_1 \to P^*(G_2)$ be a set-valued homomorphism with kernel K. Then T is a homomorphism from the group G_1 onto the group $T^*(G_1)$, where $T^{*}(G_1)$ is the set of all subsets of the form $T(g_1)$ with $g_1 \in G_1$. For a subgroup $\bar{H}$ of $T^{*}(G_1)$, let H be defined by, $H = \{x \in G_1 \mid T(x) \in \bar{H}\}$. Then H is a subgroup of G_1 and $H \supset K$; if $\bar{H}$ is a normal in $T^{*}(G_1)$ then H is normal in G_1. Moreover, this association setup a one to one mapping from the set of all subgroups of $T^{*}(G_1)$ onto the set of all subgroups of G_1 which contains K.

Proof. Let $T: G_1 \to T^{*}(G_1) \subseteq P^{*}(G_2)$, $\bar{H}$ a normal subgroup of $T^{*}(G_1)$ and let $H = \{x \in G_1 \mid T(x) \in \bar{H}\}$. If $x \in K$ then $T(x) = T(e_1) \in \bar{H}$, since $T(e_1)$ is the identity element of $T^{*}(G_1)$ and $\bar{H}$ is a subgroup of $T^{*}(G_1)$, which implies that $x \in H$. Thus, $K \subseteq H$. Now let $x, y \in H$ then $T(x), T(y) \in \bar{H}$, which implies that $T(xy) = T(x)T(y) \in \bar{H}$. Therefore, $xy \in H$. Also, if $x \in H$, then $T(x^{-1}) = (T(x))^{-1} \in \bar{H}$. That is, $x^{-1} \in H$. Hence H is a subgroup of G_1. Next if $g_1 \in G_1, h \in H$, then $T(g_1^{-1} h g_1) = T(g_1^{-1}) T(h) T(g_1) \in \bar{H}$, since $T(h) \in \bar{H}$ and $T(g_1) \in T^{*}(G_1)$ and $\bar{H}$ is a normal subgroup of $T^{*}(G_1)$. Therefore, $g_1^{-1} h g_1 \in H$. Thus we have proved that, H is a normal subgroup of G_1. Since $K \subseteq H$, the set valued homomorphism $T: G_1 \to P^{*}(G_2)$, induces a homomorphism of H onto $\bar{H}$. Thus, $\frac{H}{K} \cong \bar{H}$. Conversely, let L be a subgroup of G_1 with $K \subseteq L$ and let $\bar{L} = \{A \in P^{*}(G_2) \mid A = T(\ell), \text{ for some } \ell \in L\}$. Then it can be seen that $\bar{L}$ is a sub group of $T^{*}(G_1)$. Let $H = \{y \in G_1 \mid T(y) \in \bar{L}\}$. Then, clearly, $L \subseteq H$, since $T(\ell) \in \bar{L}$ for some $\ell \in L$. On the other hand if $h \in H$ then $T(h) \in \bar{L}$. By definition of $\bar{L}$, $T(h) = T(\ell)$, for some $\ell \in L$. Thus, $T(h\ell^{-1}) = T(h)T(\ell^{-1}) = T(h)[T(\ell)]^{-1} = T(\ell)[T(\ell)]^{-1} = T(\ell\ell^{-1}) = T(e_1)$. That is, $h\ell^{-1} \in K \subseteq L$. Therefore, $h \in L.l = L$. Hence $H \subseteq L$, which implies that $H = L$.

Theorem 4.8 Let $T: G_1 \to P^{*}(G_2)$ be a set-valued homomorphism so that $T: G_1 \to T^{*}(G_1)$ is an onto set-valued homomorphism. Let $\bar{N}$ be a normal subgroup of $T^{*}(G_1)$ and let $N = \{x \in G_1 \mid T(x) \in \bar{N}\}$. Then $\frac{G_1}{N} \cong \frac{T^{*}(G_1)}{\bar{N}}$. Equivalently, $\frac{G_1}{N} \cong \frac{G_1 / K}{N / K}$.

Proof. Define a mapping $\psi : G_1 \to \frac{T^*(G_1)}{\bar{N}}$ by $\psi(g_1) = \bar{N}T(g_1), \forall g_1 \in G_1$. Let $T(g_1) \in T^*(G_1)$. Since T is onto from G_1 to $T^*(G_1)$, a typical element $\bar{N}T(g_1)$ of $\frac{T^*(G_1)}{\bar{N}}$ is of the form $\bar{N}T(g_1) = \psi(g_1)$. Hence, ψ is onto.

Now, we have $\psi(g_1 g_1') = \bar{N}T(g_1 g_1') = \bar{N}T(g_1)T(g_1')$

$= \bar{N}T(g_1)\bar{N}T(g_1')$, since $\bar{N}$ is a normal subgroup of $T^*(G_1)$.

$= \psi(g_1)\psi(g_1')$, for all $g_1, g_1' \in G_1$.

Hence ψ is a homomorphism.

Also, $Ker\psi = \{g_1 \in G_1 \mid \psi(g_1) = \bar{N}\} = \{g_1 \in G_1 \mid \bar{N}T(g_1) = \bar{N}\}$

$= \{g_1 \in G_1 \mid T(g_1) \in \bar{N}\} = N.$

Hence, $\frac{G_1}{N} \cong \frac{T^*(G_1)}{\bar{N}}$. By Theorem 4.6, we have $T^*(G_1) \cong \frac{G_1}{K}$ and by Theorem 4.7, $\bar{N} = \frac{N}{K}$. Thus, $\frac{G_1}{N} \cong \frac{G_1 / K}{N / K}$.

5 Conclusion

Rough sets are the modern tool for modeling the incomplete information system. The group is of the most basic algebraic system in which many of the physical and real world problems are modeled. So, we have considered the roughness in groups with respect to a set-valued homomorphism T from a group G_1 to the set of all non-empty subsets of another group G_2. We established the connection between lower and upper approximations of normal subgroups G_1 and G_2. We also proved the isomorphism theorem for the set valued homomorphism of groups. The isomorphism theorem we have proved for groups can be extended to any algebraic systems. So, we certainly believe that this paper will generate interest among researchers to extend the results of classical algebraic systems to the rough algebraic systems.

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