A Fundamental Theorem of Set-valued Homomorphism of Groups

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Abstract

In this paper, we study some properties of lower and upper T-rough normal subgroups based on a set valued homomorphism T from a group $G_1$ to the set of all non-empty subsets of a group $G_2$. We also, prove the isomorphism theorem for lower and upper T-rough subgroups of $G_1$ and $G_2$. We introduce the concept of the kernel of the set-valued homomorphism and proved that it is a normal subgroup of $G_1$. As a main result of this paper, we prove the analog of the fundamental theorem of homomorphism of groups to the set valued homomorphism of groups.

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1 Introduction

Z. Pawlak [20], in his pioneering paper, introduced the theory of rough sets as a tool to model uncertainty, vague and incomplete information system. Since then, the theory of rough sets attracted many researchers in
mathematics, computer science and engineering. Originally rough sets are described as lower and upper approximations based on equivalence relations. Later, generalized rough sets were considered based on arbitrary relations by W. Zhu [27].

Algebraic systems on rough sets were considered by mathematicians like R. Biswas, S. Nanda, N. Kuroki, P. P. Wang, B. Davvaz and many others. R. Biswas and S. Nanda [1] introduced the concept of rough groups and rough subgroups. N. Kuroki [15] introduced the notion of rough ideal in a semigroup. N. Kuroki and P. P. Wang [16] studied the lower and upper approximations in a fuzzy group. Rough sets and rough groups were also considered by N. Kuroki and J. N. Moderson [17]. B. Davvaz have introduced roughness in many algebraic systems. Considering ring as a universal set, B. Davvaz [3] introduced the notion of rough ideals and rough subrings with respect to an ideal of the ring. In [12], O. Kazanci and B. Davvaz introduced the notion of rough prime ideals and rough primary ideals in a ring. The authors V. Selvan and G. Senthil Kumar [23, 24] have studied the roughness of ideals and fuzzy ideals in a semiring. In [7], B. Davvaz introduced the concept of set valued homomorphism and T-rough sets in a group. In [26], S. Yamak, O. Kazanci and B. Davvaz introduced the generalized lower and upper approximations in a ring based on set valued homomorphism of rings. S. B. Hosseini et al [8,9] have studied some properties of T-rough sets in semigroups, commutative rings.

In this paper, we consider the set-valued homomorphism on groups and study some properties of it. In [7], B. Davvaz proved that if \( T: G_1 \rightarrow P^*(G_2) \) is a set-valued homomorphism from a group \( G_1 \) to the set of all non-empty subsets of a group \( G_2 \), then \( H \) is a normal subgroup of \( G_2 \), then the upper approximation of \( H \), viz, \( U_r(H) \) is a normal subgroup of \( G_1 \). In section 3 of this paper, assuming \( H \) a normal subgroup of \( G_2 \) containing \( T(e_1) \), where \( e_1 \) is the identity element of \( G_1 \), we prove that the lower approximation of \( H \), viz, \( L_r(H) \) is also a normal subgroup of \( G_1 \). Also, if \( f: G_1 \rightarrow G_2 \) is a homomorphism from a group \( G_1 \) onto a group \( G_2 \) and if \( T_2: G_2 \rightarrow P^*(G_2) \) is a set-valued homomorphism then \( T_1: G_1 \rightarrow P^*(G_1) \) defined by \( T_1(x) = \left\{ g_1 \in G_1 \mid f(g_1) \in T_2(f(x)) \right\} \), \( \forall x \in G_1 \) is a set-valued homomorphism and if \( H \) is a normal subgroup of \( G_1 \) containing \( T(e_1) \) we further prove that \( G_1/L_r(H) \cong G_2/L_r(f(H)) \) and \( G_1/U_r(H) \cong G_2/U_r(f(H)) \). In section 4, we introduce the concept of kernel of a set-valued homomorphism \( T: G_1 \rightarrow P^*(G_2) \) and prove that it is a normal subgroup of \( G_1 \). As a main result of this paper, we prove the analog of the fundamental theorem of homomorphism of groups for the set-valued homomorphism.
2 Preliminaries

The set-valued mappings, the lower and upper approximation with respect to a set-valued mapping and the set-valued homomorphism of groups are introduced by B. Davvaz and can be found in [7]. We recall them for the sake of completeness.

**Definition 2.1** [7] Let $X$ and $Y$ be two non-empty sets and $B \subseteq Y$. Let $T: X \rightarrow P^r(Y)$ be a set-valued mapping, where $P^r(Y)$ denotes the set of all non-empty subsets of $Y$. The lower and upper approximations of $B$ under $T$ is given by $L_T(B) = \{x \in X \mid T(x) \subseteq B\}$; $U_T(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\}$.

**Definition 2.2** [7] Let $X$ and $Y$ be two non-empty sets and $B \in P^r(Y)$. Let $T: X \rightarrow P^r(Y)$ be a set-valued mapping, then the pair $(L_T(B), U_T(B))$ is referred to as the generalized rough set of $B$ induced by $T$.

**Definition 2.3** [7] A set-valued homomorphism $T$ from a group $G_1$ to a group $G_2$ is a mapping from $G_1$ into $P^r(G_2)$ that preserves the group operation, that is, $T(ab) = T(a)T(b)$ and $(T(a))^{-1} = \{x^{-1} \mid x \in T(a)\} = T(a^{-1})$, for all $a, b \in G_1$. 

**Example 2.4** Every homomorphism of groups can be considered as a set-valued homomorphism of groups. For, if $f: G_1 \rightarrow G_2$ is a homomorphism of groups then $T_f: G_1 \rightarrow P^r(G_2)$ defined by $T_f(x) = \{f(x)\}, \forall x \in G_1$ is a set-valued homomorphism from $G_1$ to $P^r(G_2)$.

For more examples of set-valued homomorphism of groups, one can refer [7].

**Definition 2.5** [7] Let $G$ be a group. Let $\theta$ be a congruence of $G$, that is, $\theta$ is an equivalence relation on $G$ such that $(a, b) \in \theta$ implies $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$ for all $x \in G$. We denote by $[a]_\theta$ the $\theta$-congruence class containing the element $a \in G$.

**Example 2.6** Let $G$ be a group, $H$ a normal subgroup of $G$. For $a, b \in G$, we define $a \equiv b \pmod{H}$ if and only if $ab^{-1} \in H$. Then the relation $\equiv$ is congruence relation on $G$.

**Proposition 2.7** [7] Let $\theta$ be a congruence on a group $G$. If $a, b \in G$ then

(i) $[a]_\theta [b]_\theta = [ab]_\theta$ and (ii) $[a^{-1}]_\theta = [[a]_\theta]^{-1}$.

**Corollary 2.8** [7] Let $\theta$ be a congruence on a group $G$. Define $T: G \rightarrow P^r(G)$ by $T(x) = [x]_\theta, \forall x \in G$. Then $T$ is a set-valued homomorphism.
Definition 2.9 [7]  Let $T$ be a set-valued mapping from $G_1$ into $\mathcal{P}^*(G_2)$. The mapping $T$ is said to be lower semi-uniform if for each subgroup $B$ in $G_2$, the set $L_T(B)$ is a subgroup of $G_1$. The mapping $T$ is said to be upper semi-uniform if for each subgroup $B$ in $G_2$, the set $U_T(B)$ is a subgroup of $G_1$. A set-valued mapping $T$ is said to be uniform if it is upper and lower semi-uniform.

Theorem 2.10 [7]  Every set-valued homomorphism is uniform.

Definition 2.11  Let $G_1, G_2$ be two groups, $T : G_1 \rightarrow \mathcal{P}^*(G_2)$ be a set-valued homomorphism and $H$ be a normal subgroup of $G_2$. If $L_T(H)$ is a normal subgroup of $G_1$, then $H$ is called the lower $T$-rough normal subgroup of $G_2$ and if $U_T(H)$ is a normal subgroup of $G_1$ then $H$ is called the upper $T$-rough normal subgroup of $G_2$. If $L_T(H)$ and $U_T(H)$ are normal subgroup of $G_1$, then we call $(L_T(H), U_T(H))$ a $T$-rough normal subgroup.

3 Isomorphism theorem for $T$-rough Groups

Let $G_1, G_2$ be two groups, $H$ be a normal subgroups of $G_2$ and $T : G_1 \rightarrow \mathcal{P}^*(G_2)$ be a set-valued homomorphism. In [7, Theorem 3.11] B. Davvaz proved that $U_T(H)$ is a normal subgroup of $G_1$. In the following theorem, assuming $H$ a normal subgroup of $G_2$ containing $T(e_1)$, where $e_1$ is the identity element of $G_1$, we prove that $L_T(H)$ is also a normal subgroup of $G_1$.

Theorem 3.1  Let $G_1, G_2$ be two groups, $H$ be a normal subgroup of $G_2$ containing $T(e_1)$ and $T : G_1 \rightarrow \mathcal{P}^*(G_2)$ be a set-valued homomorphism. Then $L_T(H)$ is a normal subgroup of $G_1$.

Proof. By [7] Theorem [4.10], if $H$ is a subgroup of $G_2$ then $L_T(H)$ is a subgroup of $G_1$. Suppose $g_1 \in G_1$ and $x \in L_T(H)$ then $T(x) \subseteq H$.

$$T(g_i^{-1} x g_1) = T(g_i^{-1}) T(x) T(g_1) \subseteq T(g_i^{-1}) H T(g_1)$$
$$= T(g_i^{-1}) T(g_1) H, \text{ since } H \text{ is normal in } G_2$$
$$= T(g_i^{-1} g_1) H = T(e_1) H = H.$$ That is, $g_i^{-1} x g_1 \in L_T(H)$, which proves that $L_T(H)$ is a normal subgroup of $G_1$. 

Corollary 3.2 Let $G_1$ and $G_2$ be two groups, $T : G_1 \rightarrow P^\ast (G_2)$ be a set-valued homomorphism and $H$ a normal subgroup of $G_2$ containing $T(e_1)$, then $H$ is a $T$-rough normal subgroup.

Theorem 3.3 Let $G_1$ and $G_2$ be two groups, $f : G_1 \rightarrow G_2$ be an epimorphism and $T_2 : G_2 \rightarrow P^\ast (G_2)$ be a set-valued homomorphism. If $f$ is one-to-one and $T_i(x) = \{ g_i \in G_i \mid f(g_i) \in T_2(f(x)) \}, \forall x \in G_i$ then $T_i : G_1 \rightarrow P^\ast (G_1)$ is a set-valued homomorphism.

Proof. Let $u \in T_i(xy)$. Then $f(u) \in T_2(f(xy)) = T_2(f(x)f(y))$. That is $f(u) = ab$, for some $a \in T_2(f(x)), b \in T_2(f(y))$. Since $f$ is onto, there exists $c, d \in G_i$ such that $f(c) = a, f(d) = b$. Hence, $f(u) = f(c)f(d), c \in T_i(x)$ and $d \in T_i(y)$. Therefore, $u = cd$, which implies that $u \in T_i(xy)$. Hence, $T_i(x)T_i(y) \subseteq T_i(xy)$. Conversely, assume that $z \in T_i(x)T_i(y)$. Then $z = cd$ for some $c \in T_i(x)$ and $d \in T_i(y)$ and so, $f(c) \in T_2(f(x)), f(d) \in T_2(f(y))$. Therefore, $f(cd) = f(c)f(d) \in T_2(f(x))T_2(f(y)) = T_2(f(x)f(y)) = T_2(f(xy))$ which implies that, $z = cd \in T_i(xy)$. Therefore, $T_i(x)T_i(y) \subseteq T_i(xy)$. Hence, $T_i(xy) = T_i(x)T_i(y)$.

We have, $c \in T_i(x^{-1}) \Leftrightarrow f(c) \in T_2(f(x^{-1})) \Leftrightarrow f(c) \in T_2((f(x))^{-1}) 
\Leftrightarrow f(c) \in T_2(f(x))^{-1} \Leftrightarrow (f(c))^{-1} \in T_2(f(x)) \Leftrightarrow f(c^{-1}) \in T_2(f(x)) 
\Leftrightarrow c^{-1} \in T_i(x) \Leftrightarrow c \in (T_i(x))^{-1}$. Therefore, $T_i(x^{-1}) = (T_i(x))^{-1}, \forall x \in G_i$.

Thus, $T_i$ is a set-valued homomorphism.

Theorem 3.4 Let $G_1$ and $G_2$ be two groups, $f : G_1 \rightarrow G_2$ be an onto homomorphism and let $T_2 : G_2 \rightarrow P^\ast (G_2)$ be a set-valued homomorphism. If $T_i(x) = \{ g_i \in G_i \mid f(g_i) \in T_2(f(x)) \}, \forall x \in G_i$ and $A$ is a non-empty subset of $G_1$, then

(i) $f(U_{T_i}(A)) = U_{T_i}(f(A))$
(ii) $f(L_{T_i}(A)) \subseteq L_{T_i}(f(A))$. Moreover, if $f$ is one-to-one then equality holds.

Proof. (i) Let $y \in f(U_{T_i}(A))$. Then there exists $x \in U_{T_i}(A)$ such that $y = f(x)$. As $x \in U_{T_i}(A), T_i(x) \cap A \neq \emptyset$. Hence, there exists $a \in A$ with
$a \in T_1(x)$. Thus $f(a) \in T_2(f(x))$ and $f(a) \in f(A)$, which implies that $T_2(f(x)) \cap f(A) \neq \emptyset$. Therefore, $y = f(x) \in U_{T_1}(f(A))$. Conversely, if $y \in U_{T_1}(f(A))$, then since $f$ is onto, there exists $x \in G_1$ such that $y = f(x) \in U_{T_1}(f(A))$. That is, $T_2(f(x)) \cap f(A) \neq \emptyset$. Hence there exists $z \in T_2(f(x)) \cap f(A)$. Thus, $z = f(a)$ for some $a \in A$. That is, $z = f(a) \in T_2(f(x))$. Hence, $a \in T_1(x) \cap A$, so that $T_1(x) \cap A \neq \emptyset$. This proves that $x \in U_{T_1}(A)$ and $y = f(x) \in f(U_{T_1}(A))$. Hence, $U_{T_1}(f(A)) \subseteq f(U_{T_1}(A))$. Thus $f(U_{T_1}(A)) = U_{T_1}(f(A))$.

(ii) Let $y \in f(L_{T_1}(A))$. Then there exists $x \in L_{T_1}(A)$ such that $y = f(x)$. Now, let $w \in T_2(f(x))$. Since $f$ is onto, there exists $z \in G_1$ such that $f(z) = w$. Thus, $w = f(z) \in T_2(f(x))$, which implies that $z \in T_1(x)$. As $x \in L_{T_1}(A)$, it follows that $z \in T_1(x) \subseteq A$. Thus, $w = f(z) \in f(A)$. Hence, $T_2(f(x)) \subseteq f(A)$ which proves that $y = f(x) \in L_{T_1}(f(A))$. Conversely, let $y \in L_{T_1}(f(A))$. Then $T_2(y) \subseteq f(A)$. Since $f$ is onto, there exists $x \in G_1$, such that $y = f(x)$. Therefore, $T_2(f(x)) \subseteq f(A)$. Now, let $u \in T_1(x)$. Then $f(u) \in T_2(f(x)) \subseteq f(A)$. Thus, there exists $a \in A$ such that $f(u) = f(a)$. Since $f$ is an one-to-one, $u = a \in A$. Therefore, $T_1(x) \subseteq A$. That is, $x \in L_{T_1}(A)$. Hence $y = f(x) \in f(L_{T_1}(A))$, which proves that $L_{T_2}(f(A)) \subseteq f(L_{T_1}(A))$. Thus, $f(L_{T_1}(A)) = L_{T_2}(f(A))$.

**Theorem 3.5** Let $G_1$ and $G_2$ be two groups, $f$ be an isomorphism from $G_1$ onto $G_2$ and $T_2 : G_2 \rightarrow P(G_1)$ be a set-valued homomorphism. Let $H$ be a subgroup of $G_1$. If $T_1(x) = \{ g_1 \in G_1 \mid f(g_1) \in T_2(f(x)) \}, \forall x \in G_1$, then

(i) $U_{T_1}(H)$ is a subgroup of $G_1$ if and only if $U_{T_1}(f(H))$ is a subgroup of $G_2$.

(ii) $U_{T_1}(H)$ is a normal subgroup of $G_1$ if and only if $U_{T_1}(f(H))$ is a normal subgroup of $G_2$.

(iii) $L_{T_1}(H)$ is a subgroup of $G_1$ if and only if $L_{T_1}(f(H))$ is a subgroup of $G_2$.

(iv) $L_{T_1}(H)$ is a normal subgroup of $G_1$ if and only if $L_{T_1}(f(H))$ is a normal subgroup of $G_2$. 
**Set-valued homomorphism of groups**

**Proof.** Since $f : G_1 \to G_2$ is an isomorphism, a subset $A$ is a subgroup of $G_1$ iff $f(A)$ is a subgroup of $G_2$. Hence the proof follows from Theorem 3.4.

**Theorem 3.6** Let $G_1$ and $G_2$ be two groups, $f$ be an isomorphism from $G_1$ onto $G_2$ and let $T_2 : G_2 \to P^*(G_2)$ be a set-valued homomorphism. If $T_1(x) = \{g_1 \in G_1 \mid f(g_1) \in T_2(f(x))\}, \forall x \in G_1$ and $A$ a normal subgroup of $G_1$ containing $T_1(e)$ then

$$
(i) \quad \frac{G_1}{L_{T_1}(A)} \cong \frac{G_2}{L_{T_2}(f(A))}
$$

and

$$
(ii) \quad \frac{G_1}{U_{T_1}(A)} \cong \frac{G_2}{U_{T_2}(f(A))}
$$

**Proof.** Define $\phi : G_1 \to \frac{G_2}{L_{T_2}(f(A))}$ by $\phi(g_1) = L_{T_2}(f(A))f(g_1)$, where $g_1 \in G_1$

Clearly, $\phi$ is well defined.

Let $g_1, g_1' \in G_1$. Then

$$
\phi(g_1g_1') = L_{T_2}(f(A))f(g_1g_1') = L_{T_2}(f(A))f(g_1)f(g_1')
$$

since $f$ is a homomorphism

$$
= L_{T_2}(f(A))f(g_1)L_{T_2}(f(A))f(g_1')
$$

since $L_{T_2}(f(A))$ is normal in $G_2$

$$
= \phi(g_1)\phi(g_1')
$$

Therefore, $\phi$ is homomorphism.

Ker $\phi = \{g_1 \in G_1 \mid \phi(g_1) = L_{T_2}(f(A))\}$

$$
= \{g_1 \in G_1 \mid L_{T_2}(f(A))f(g_1) = L_{T_2}(f(A))\}
$$

$$
= \{g_1 \in G_1 \mid f(g_1) \in L_{T_2}(f(A))\} = \{g_1 \in G_1 \mid g_1 \in L_{T_1}(A)\} = L_{T_1}(A)
$$

Let $L_{T_1}(f(A))g_2 \in \frac{G_2}{L_{T_2}(f(A))}$. Since $f$ is onto, there exists $g_1 \in G_1$ such that $f(g_1) = g_2$. Hence, $L_{T_2}(f(A))g_2 = L_{T_2}(f(A)f(g_1)) = \phi(g_1)$. Therefore, $\phi$ is onto.

By fundamental theorem of homomorphism of groups,

$$
\frac{G_1}{\text{ker } \phi} \cong \frac{G_2}{L_{T_2}(f(A))}.
$$

That is,

$$
\frac{G_1}{L_{T_1}(A)} \cong \frac{G_2}{L_{T_2}(f(A))}.
$$

The proof of (ii) is similar to the proof of (i).
4 Isomorphism theorem for set-valued homomorphism of Groups

In this section, we introduce the concept of the kernel of set-valued homomorphism of groups and establish the isomorphism theorems of groups in the context of set-valued homomorphism of groups.

Let $G_1$ and $G_2$ be two groups and $T: G_1 \rightarrow P^*(G_2)$ be a set-valued homomorphism. Define $K = \{ x \in G_1 | T(x) = T(e_1) \}$, where $e_1$ is the identity element of $G_1$. $K$ is called the kernel of $T$.

**Theorem 4.1** $K$ is a normal subgroup of $G_1$.

**Proof.** Let $x, y \in K$. Then $T(x) = T(e_1)$ and $T(y) = T(e_1)$. Since $T$ is a set-valued homomorphism, $T(xy) = T(x)T(y) = T(e_1)T(e_1) = T(e_1)$. Therefore, $xy \in K$. Also, $T(x^{-1}) = [T(x)]^{-1} = [T(e_1)]^{-1} = T(e_1^{-1}) = T(e_1)$, which proves that $x^{-1} \in K$. Now, if $g_1 \in G_1$ and $x \in K$ then $T(g_1xg_1^{-1}) = T(g_1)T(x)T(g_1^{-1}) = T(g_1)T(e_1)T(g_1^{-1}) = T(g_1, e_1, g_1^{-1}) = T(e_1)$. That is, $g_1xg_1^{-1} \in K$. Hence, $K$ is a normal subgroup of $G_1$.

**Theorem 4.2** Let $G_1$ and $G_2$ be two groups. If $T: G_1 \rightarrow P^*(G_2)$ is a set-valued homomorphism then $T(G_1) = \cup_{g \in G_1} T(g_1)$ is a subgroup of $G_2$.

**Proof.** Let $y_1, y_2 \in \cup_{g \in G_1} T(g_1)$. Then there exists $g_1, g_1' \in G_1$ such that $y_1 \in T(g_1)$ and $y_2 \in T(g_1')$. Then $y_1y_2 \in T(g_1)T(g_1') = T(g_1g_1') \subseteq \cup_{g \in G_1} T(g_1)$. Therefore, $y_1y_2 \in T(G_1)$ and $y_1^{-1} \in T(g_1)^{-1} = T(g_1^{-1}) \subseteq \cup_{g \in G_1} T(g_1)$. Hence, $T(G_1)$ is a subgroup of $G_2$.

**Lemma 4.3** Let $G_1$ and $G_2$ be two groups and $T: G_1 \rightarrow P^*(G_2)$ a set-valued homomorphism, then $e_2 \in T(e_1)$, where $e_1, e_2$ are the identity elements of $G_1$ and $G_2$ respectively.

**Proof.** Let $g_1 \in G_1$, and let $T(g_1) = \{ x_1, x_2, ..., x_k \} \subseteq G_2$, then $[T(g_1)]^{-1} = \{ x_1^{-1}, x_2^{-1}, ..., x_k^{-1} \} \subseteq G_2$. We have $e_2 = x_1x_1^{-1} \in T(g_1)T(g_1^{-1}) = T(g_1, g_1^{-1}) = T(e_1)$. That is, $e_2 \in T(e_1)$. 

Lemma 4.4 Let $G_1$ and $G_2$ be two groups and $T: G_1 \rightarrow P^*(G_2)$ a set-valued homomorphism. For $A \subseteq G_1$, define $T(A) = \bigcup_{a \in A} T(a)$. If $A$ is a subgroup of $G_1$, then $T(A)$ is a subgroup of $G_2$.

**Proof.** Let $x_1, x_2 \in G_2$ such that $x_1, x_2 \in T(A)$. Then $x_1 \in T(a_1)$ and $x_2 \in T(a_2)$, for some $a_1, a_2 \in A$. Thus, $x_1 x_2 \in T(a_1) T(a_2) = T(a_1 a_2) \subseteq T(A)$, and $x_1^{-1} \in T(a_1)^{-1} = T(a_1^{-1}) \subseteq T(A)$. Hence, $T(A)$ is a subgroup of $G_2$.

**Corollary 4.5** $T(e_i)$ is a subgroup of $G_2$.

If $T: G_1 \rightarrow P^*(G_2)$ is a set valued homomorphism then define $T^*(G_1) = \{T(g_i) \mid g_i \in G_1\}$. It can be easily seen that $T^*(G_1)$ is a group under the operation given by $T(g_i) * T(g_i') = T(g_i g_i')$.

**Theorem 4.6** Let $G_1$ and $G_2$ be two groups and $T: G_1 \rightarrow P^*(G_2)$ be a set-valued homomorphism with kernel $K$. Then $\frac{G_1}{K} \cong T^*(G_1)$.

**Proof.** Define a map $\psi : \frac{G_1}{K} \rightarrow T^*(G_1)$ by $\psi(g_i K) = T(g_i)$, where $g_i \in G_1$.

Suppose $g_i K = g_i' K$. Then $g_i g_i'^{-1} \in K \Rightarrow T\left(g_i g_i'^{-1}\right) = T(e_i)$

$\Rightarrow T(g_i) T\left(g_i'^{-1}\right) = T(e_i) \Rightarrow T\left(g_i\right) T\left(g_i')\right)^{-1} = T(e_i) \Rightarrow T\left(g_i\right) = T\left(g_i'\right)$

$\Rightarrow \psi(g_i K) = \psi\left(g_i' K\right)$

Hence, $\psi$ is well defined.

Clearly $\psi$ is onto, since the pre image of $T(g_i) \in T^*(G_1)$ is $g_i K$. That is, $\psi(g_i K) = T(g_i)$. Next, let $g_i, g_i' \in G_1$, then $\psi\left(g_i K \cdot g_i' K\right) = \psi\left(g_i g_i' K\right)$

$= T\left(g_i g_i'\right) = \psi\left(g_i K\right) \psi\left(g_i' K\right)$. Therefore, $\psi$ is homomorphism.

Now, suppose $\psi\left(g_i K\right) = \psi\left(g_i' K\right) \Rightarrow T\left(g_i\right) = T\left(g_i'\right) \Rightarrow T\left(g_i\right) T\left(g_i'^{-1}\right)$

$= T(e_i) \Rightarrow T\left(g_i g_i'^{-1}\right) = T(e_i) \Rightarrow g_i g_i'^{-1} \in K \Rightarrow g_i K = g_i' K$.

Therefore, $\psi$ is one-to-one. Thus, we have proved that $\frac{G_1}{K} \cong T^*(G_1)$.

**Theorem 4.7** Let $T: G_1 \rightarrow P^*(G_2)$ be a set-valued homomorphism with kernel $K$. Then $T$ is a homomorphism from the group $G_1$ onto the group $T^*(G_1)$,
where \( T^*(G_i) \) is the set of all subsets of the form \( T(g_i) \) with \( g_i \in G_i \). For a subgroup \( \overline{H} \) of \( T^*(G_i) \), let \( H \) be defined by, \( H = \{ x \in G_i \mid T(x) \in \overline{H} \} \). Then \( H \) is a subgroup of \( G_i \) and \( H \supseteq K \); if \( \overline{H} \) is a normal in \( T^*(G_i) \) then \( H \) is normal in \( G_i \). Moreover, this association setup a one to one mapping from the set of all subgroups of \( T^*(G_i) \) onto the set of all subgroups of \( G_i \) which contains \( K \).

**Proof.** Let \( T : G_i \rightarrow T^*(G_i) \subseteq P^*(G_j) \), \( \overline{H} \) a normal subgroup of \( T^*(G_i) \) and let \( H = \{ x \in G_i \mid T(x) \in \overline{H} \} \). If \( x \in K \) then \( T(x) = T(e_i) \in \overline{H} \), since \( T(e_i) \) is the identity element of \( T^*(G_i) \) and \( \overline{H} \) is a subgroup of \( T^*(G_i) \), which implies that \( x \in H \). Thus, \( K \subseteq H \). Now let \( x, y \in H \) then \( T(x), T(y) \in \overline{H} \), which implies that \( T(xy) = T(x)T(y) \in \overline{H} \). Therefore, \( xy \in H \). Also, if \( x \in H \), then \( T(x^{-1}) = (T(x))^{-1} \in \overline{H} \). That is, \( x^{-1} \in H \). Hence \( H \) is a subgroup of \( G_i \). Next if \( g_i \in G_i, h \in H \), then \( T(g_i^{-1}h) = T(g_i^{-1})T(h)T(g_i) \in \overline{H} \), since \( T(h) \in \overline{H} \) and \( T(g_i) \in T^*(G_i) \) and \( \overline{H} \) is a normal subgroup of \( T^*(G_i) \). Therefore, \( g_i^{-1}h \in H \). Thus we have proved that, \( H \) is a normal subgroup of \( G_i \). Since \( K \subseteq H \), the set valued homomorphism \( T : G_i \rightarrow P^*(G_j) \), induces a homomorphism of \( H \) onto \( \overline{H} \). Thus, \( \frac{H}{K} \cong \overline{H} \). Conversely, let \( L \) be a subgroup of \( G_i \) with \( K \subseteq L \) and let \( \overline{L} = \{ A \in P^*(G_j) \mid A = T(\ell) \text{ for some } \ell \in L \} \). Then it can be seen that \( \overline{L} \) is a subgroup of \( T^*(G_i) \). Let \( H = \{ y \in G_i \mid T(y) \in \overline{L} \} \). Then, clearly, \( L \subseteq H \), since \( T(\ell) \in \overline{L} \) for some \( \ell \in L \). On the other hand if \( h \in H \) then \( T(h) \in \overline{L} \). By definition of \( \overline{L} \), \( T(h) = T(\ell) \), for some \( \ell \in L \). Thus, \( T(h\ell^{-1}) = T(h)T(\ell^{-1}) = T(h)[T(\ell)]^{-1} = T(\ell)[T(\ell^{-1})]^{-1} = T(\ell^{-1}) = T(e_i) \). That is, \( h\ell^{-1} \in K \subseteq L \). Therefore, \( h \in LL = L \). Hence \( H \subseteq L \), which implies that \( H = L \).

**Theorem 4.8** Let \( T : G_i \rightarrow P^*(G_j) \) be a set-valued homomorphism so that \( T : G_i \rightarrow T^*(G_i) \) is an onto set-valued homomorphism. Let \( \overline{N} \) be a normal subgroup of \( T^*(G_i) \) and let \( N = \{ x \in G_i \mid T(x) \in \overline{N} \} \). Then \( \frac{G_i}{N} \cong \frac{T^*(G_i)}{\overline{N}} \). Equivalently, \( \frac{G_i}{N} \cong \frac{G_i}{K}{\frac{N}{K}} \).
**Proof.** Define a mapping $\psi : G_i \rightarrow \frac{T^*(G_i)}{N}$ by $\psi(g_i) = \overline{NT(g_i)}, \forall g_i \in G_i$.

Let $T(g_i) \in T^*(G_i)$. Since $T$ is onto from $G_i$ to $T^*(G_i)$, a typical element $\overline{NT(g_i)}$ of $\frac{T^*(G_i)}{N}$ is of the form $\overline{NT(g_i)} = \psi(g_i)$. Hence, $\psi$ is onto.

Now, we have $\psi(g_ig_i') = \overline{NT(g_ig_i')} = \overline{NT(g_i)T(g_i')}$
$$= \overline{NT(g_i)} \overline{NT(g_i')}, \text{ since } \overline{N} \text{ is a normal subgroup of } T^*(G_i).$$
$$= \psi(g_i)\psi(g_i'), \text{ for all } g_i, g_i' \in G_i.$$ Hence $\psi$ is a homomorphism.

Also, $\ker \psi = \{ g_i \in G_i \mid \psi(g_i) = \overline{N} \} = \{ g_i \in G_i \mid \overline{NT(g_i)} = \overline{N} \}$
$$= \{ g_i \in G_i \mid T(g_i) \in \overline{N} \} = N.$$

Hence, $\frac{G_i}{N} \cong \frac{T^*(G_i)}{N}$. By Theorem 4.6, we have $T^*(G_i) \cong \frac{G_i}{K}$
and by Theorem 4.7, $\overline{N} = \frac{N}{K}$. Thus, $\frac{G_i}{N} \cong \frac{G_i / K}{N / K}$.

## 5 Conclusion

Rough sets are the modern tool for modeling the incomplete information system. The group is of the most basic algebraic system in which many of the physical and real world problems are modeled. So, we have considered the roughness in groups with respect to a set-valued homomorphism $T$ from a group $G_i$ to the set of all non-empty subsets of another group $G_j$. We established the connection between lower and upper approximations of normal subgroups $G_i$ and $G_j$. We also proved the isomorphism theorem for the set valued homomorphism of groups. The isomorphism theorem we have proved for groups can be extended to any algebraic systems. So, we certainly believe that this paper will generate interest among researchers to extend the results of classical algebraic systems to the rough algebraic systems.

## References


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