A Note on Disjoint Dominating Sets in Graphs

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Abstract

The disjoint domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of the union of two disjoint dominating sets in $G$. The disjoint independent domination number of a graph $G$ is the minimum cardinality of the union of two disjoint independent dominating sets in $G$. In this paper we study these two parameters. We determine the value of $\gamma(G)$ for several graphs and give partial answers to some open problems posed in [5].

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1 Introduction

By a graph we mean a finite, undirected, connected graph without loops and multiple edges. For graphs theoretical terms we refer Harary [3] and for terms related to domination we refer Haynes et al. [4].

A subset $S$ of $V$ is said to be a dominating set in $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$ and a dominating set
of minimum cardinality is called a $\gamma$-set of $G$. A set $S \subseteq V$ is said to be independent if no two vertices in $S$ are adjacent. The minimum cardinality of a maximal independent dominating set is called the independent domination number and it is denoted by $i(G)$.

In [5], S. M. Hedetniemi et al. introduced the concept of disjoint dominating sets in graphs. The disjoint domination number $\gamma(G)$ is defined as $\gamma(G) = \min\{|S_1| + |S_2| : S_1$ and $S_2$ are disjoint dominating sets of $G\}$. We say that two disjoint dominating sets whose union has cardinality $\gamma(G)$ is a $\gamma\gamma$-pair of $G$. The disjoint independent domination number $ii(G)$ is defined as the minimum cardinality of the union of two disjoint independent dominating sets in a graph $G$. In this paper we study the existence of disjoint dominating sets and disjoint independent dominating sets in graphs and find partial solutions to some problems posed in [5]. The domatic partition of $G$ is a partition of $V(G)$, all of whose classes are dominating sets in $G$. The maximum number of classes of a domatic partition of $G$ is called the domatic number of $G$ and is denoted by $d(G)$. A graph is called domatically full if $d(G) = \delta(G) + 1$ which is the maximum possible order of a domatic partition of $V$. Here $\delta(G)$ is the minimum degree of a vertex of $G$. We need the following.

**Definition 1.1** [1] The $n$-star graph $S_n$ is a simple graph whose vertex set is the set of all $n!$ permutations of $\{1, 2, 3, \ldots, n\}$ and two vertices $\alpha$ and $\beta$ are adjacent if and only if $\alpha(1) \neq \beta(1)$ and $\alpha(i) \neq \beta(i)$ for exactly one $i$, $i \neq 1$.

**Theorem 1.2** [1] $\gamma(S_n) = (n - 1)!$ and $S_n$ is domatically full.

**Proposition 1.3** [5] For any cycle $C_n$, $n \geq 3$, $\gamma\gamma(C_n) = ii(C_n) = 2\gamma(C_n) = 2 \left\lceil \frac{n}{3} \right\rceil$.

**Proposition 1.4** [5] $\gamma\gamma(P_{3k+2}) = ii(P_{3k+2}) = 2\gamma(P_{3k+2}) = 2k + 2$.

**Proposition 1.5** [5] For the complete graph $K_n$, $\gamma\gamma(K_n) = ii(K_n) = 2$.

**Theorem 1.6** [5] A connected graph $G$ is $\gamma\gamma$-maximum if and only if either $G = C_4$ or every vertex is a leaf or a stem.

## 2 Main Results

**Definition 2.1** The disjoint domination number $\gamma\gamma(G)$ is defined as $\gamma\gamma(G) = \min\{|S_1| + |S_2| : S_1$ and $S_2$ are disjoint dominating sets of $G\}$.

**Definition 2.2** The disjoint independent domination number $ii(G)$ is defined as the minimum cardinality of the union of two disjoint independent dominating sets in a graph $G$. 
We say that a graph $G$ is $\gamma\gamma$-minimum if it has two disjoint $\gamma$-sets, that is $\gamma\gamma(G)=2\gamma(G)$. Similarly a graph $G$ is called $\gamma\gamma$-maximum if $\gamma\gamma(G)=n$.

**Theorem 2.3** If $G$ is a graph with at least two universal vertices, then $\gamma\gamma(G)=\ii(G)=2$.

**Proof.** Let $u$ and $v$ be two universal vertices of the graph $G$. Then $\{u\}$ and $\{v\}$ are two disjoint independent dominating sets of $G$ and hence $\gamma\gamma(G)=\ii(G)=2$. □

**Theorem 2.4** If $W_n$ is the wheel $C_n + K_1$, $\gamma\gamma(W_n)=1+\lceil \frac{n}{3} \rceil =\ii(W_n)$.

**Proof.** The singleton set $S$ containing the center of the wheel is the $\gamma$-set of $W_n$. Hence $\gamma(W_n)=1$. $\gamma\gamma(C_n)=2\lceil \frac{n}{3} \rceil$ by Proposition 1.3 and a $\gamma$-set of the cycle of $W_n$ say $S'$ dominates center of $W_n$. Also $S$ and $S'$ are independent and $S \cap S' = \phi$. Hence $\gamma\gamma(W_n)=1+\lceil \frac{n}{3} \rceil =\ii(W_n)$. □

**Corollary 2.5** A wheel $W_n$ is a $\gamma\gamma$-minimum graph $\iff n=3$.

**Proof.** $W_n$ is complete if and only if $n=3$ and so the proof follows. □

**Definition 2.6** A graph obtained from a wheel by attaching a pendent edge at each vertex of an $n$-cycle is a helm and is denoted by $H_n$. Thus $H_n$ is a graph of order $2n+1$.

**Theorem 2.7** For a helm $H_n$

(i) $\gamma\gamma(H_n)=2n$

(ii) $\ii(H_n) = \begin{cases} 2n & \text{if } n \text{ is even} \\ \text{does not exist} & \text{if } n \text{ is odd} \end{cases}$

**Proof.** We know that the helm $H_n$ contains $2n+1$ vertices. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle, $v_1, v_2, v_3, \ldots, v_n$ be the corresponding pendent vertices and $v$ be the center. Then $S = \{u_1, u_2, u_3, u_4, \ldots, u_{n-3}, v_{n-2}, u_{n-1}, v_n\}$ and $S' = \{v_1, u_2, v_3, u_4, \ldots, v_{n-3}, u_{n-2}, v_{n-1}, u_n\}$ are two disjoint $\gamma$-sets of $H_n$. Hence $\gamma\gamma(H_n) = 2n$.

Case(i) $n$ is even

Then $S$ and $S'$ are independent sets and so $\ii(H_n) = 2n$

Case(ii) $n$ is odd

Then either $S$ or $S'$ is not independent and hence $\ii(H_n)$ does not exist. □

We note that $H_n$ is a $\gamma\gamma$-minimum graph.

**Definition 2.8** Web graph is a graph obtained by joining the pendent vertices of a helm $H_n$ to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. It is a graph of order $3n+1$. 

Theorem 2.9 For a web graph $G$,
(i) $\gamma(G) = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil$
(ii) $\iota(G) = \begin{cases} \frac{5n}{2} + 1 & \text{if } n \text{ is even} \\ \text{does not exist} & \text{if } n \text{ is odd} \end{cases}$

Proof. The web graph contains 2 cycles of order $n$, $n$ pendent vertices and a center. Thus $|V(G)| = 3n + 1$.
Claim: $\gamma(G) = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil$.
Let $S$ be the $\gamma$-set of $G$ obtained by taking the alternate vertices of the outer cycle, the alternate pendent vertices (not corresponding to the vertices taken in the outer cycle) and the center. Thus $|S| = n + 1$. The other dominating set $S'$ of $G$ can be obtained by taking the remaining vertices of the outer cycle, the remaining pendent vertices and a $\gamma$-set of the inner cycle (not corresponding to the vertices in the outer cycle). Thus $S' = n + 1 + \left\lceil \frac{n}{3} \right\rceil$. Also $S \cap S' = \phi$. Hence $\gamma(G) = n + 1 + n + \left\lceil \frac{n}{3} \right\rceil = 2n + 1 + \left\lceil \frac{n}{3} \right\rceil$.
We now determine $\iota(G)$.
Case(i): $n$ is even
The first $\gamma$-set $S$ can be obtained as above and it is independent. The other independent dominating set $S''$ can be obtained by considering the set of remaining vertices of outer cycle, remaining pendent vertices and the alternate vertices of inner cycle such that $S \cap S'' = \phi$. Therefore $|S \cup S''| = n + 1 + n + \frac{n}{2}$. Thus $\iota(G) = \frac{5n}{2} + 1$.
Case(ii): $n$ is odd
We cannot find two disjoint dominating sets. Hence $\iota(G)$ does not exist in this case.

Definition 2.10 Grid graph is the Cartesian product of two paths.

Theorem 2.11 $\gamma(P_2 \times P_n) = 2\gamma(P_2 \times P_n)$.

Proof. For $P_2 \times P_1$ and $P_2 \times P_2$, the result is obvious. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the two rows of the grid as shown in the
Proof.

Case(i): $n = 4k - 1$, $k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-2}, u_{n-1}, v_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-2}, v_{n-1}, u_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma \gamma(P_2 \times P_n) = 2 \gamma(P_2 \times P_n)$.

Case(ii): $n = 4k$, $k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, v_{n-5}, u_{n-3}, v_{n-1}, u_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, u_{n-5}, v_{n-3}, u_{n-1}, v_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma \gamma(P_2 \times P_n) = 2 \gamma(P_2 \times P_n)$.

Case(iii): $n = 4k + 1$, $k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-4}, v_{n-2}, u_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-4}, u_{n-2}, v_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma \gamma(P_2 \times P_n) = 2 \gamma(P_2 \times P_n)$.

Case(iv): $n = 4k + 2$, $k \in \mathbb{N}$

$S = \{u_1, v_3, u_5, v_7, \ldots, u_{n-5}, v_{n-3}, u_{n-1}, v_n\}$ and $S' = \{v_1, u_3, v_5, u_7, \ldots, v_{n-5}, u_{n-3}, v_{n-1}, u_n\}$ are two disjoint $\gamma$-sets of $P_2 \times P_n$ with $|S| = |S'|$. Hence $\gamma \gamma(P_2 \times P_n) = 2 \gamma(P_2 \times P_n)$.

From the above cases we get, $\gamma \gamma(P_2 \times P_n) = 2 \gamma(P_2 \times P_n)$.

\[\begin{align*}
\text{Theorem 2.12 } \gamma \gamma(P_3 \times P_n) &= \begin{cases} 
3 & \text{if } n=1 \\
4 & \text{if } n=2 \\
8 & \text{if } n=4 \\
6k & \text{if } n=4k-1 \\
2 \gamma(P_3 \times P_n) + 1 & \text{otherwise}
\end{cases}
\end{align*}\]

Proof.

When $n = 1, 2$ and $4$ the result is obvious. Let $\{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ be the vertices of the three rows of the grid
$P_3 \times P_n$, as shown in the figure.

In [6], it has been proved that $\gamma(P_3 \times P_n) = \left\lceil \frac{3n+4}{4} \right\rceil$.

Case(i): $n = 4k - 1$, $k \in \mathbb{N}$

$S = \{u_1, w_1, v_3, u_5, v_7, \ldots, u_{n-2}, w_{n-2}, v_n\}$ and $S' = \{v_1, u_3, w_3, v_5, w_7, \ldots, v_{n-2}, u_n, w_n\}$ are two disjoint $\gamma$-sets of $P_3 \times P_n$ with $|S| = |S'| = 3k$. Hence $\gamma \gamma(P_3 \times P_n) = 6k$.

Case(ii): $n = 4k$, $k \in \mathbb{N} - \{1\}$

$S = \{u_1, w_1, v_3, u_5, v_7, \ldots, u_{n-3}, w_{n-3}, v_{n-1}, u_n, w_n\}$ and $S' = \{v_1, u_3, v_3, w_5, u_7, v_7, \ldots, u_{n-5}, w_{n-5}, v_{n-3}, u_{n-1}, w_{n-1}, v_n\}$ are two disjoint $\gamma$-sets of $P_3 \times P_n$ with $|S| = \left\lfloor \frac{3k+4}{4} \right\rfloor$ and $|S'| = \left\lfloor \frac{3k+4}{4} \right\rfloor + 1$. Hence $\gamma \gamma(P_3 \times P_n) = 2\gamma(P_3 \times P_n) + 1$.

Case(iii): $n = 4k + 1$, $k \in \mathbb{N}$

$S = \{u_1, w_1, v_3, u_5, w_7, \ldots, v_{n-2}, u_n, w_n\}$ and $S' = \{v_1, u_3, v_3, w_5, u_7, v_7, \ldots, v_{n-3}, w_{n-3}, u_{n-1}, v_n\}$ are two disjoint $\gamma$-sets of $P_3 \times P_n$ with $|S| = \gamma(P_3 \times P_n)$ and $|S'| = \gamma(P_3 \times P_n) + 1$. Hence $\gamma \gamma(P_3 \times P_n) = 2\gamma(P_3 \times P_n) + 1$.

Case(iv): $n = 4k + 2$, $k \in \mathbb{N}$

$S = \{u_1, w_1, v_3, u_5, v_7, \ldots, w_{n-3}, v_{n-1}, w_n\}$ and $S' = \{v_1, u_3, w_3, v_5, u_7, v_7, \ldots, u_{n-3}, w_{n-3}, v_{n-1}, v_n\}$ are two disjoint $\gamma$-sets of $P_3 \times P_n$ with $|S| = \gamma(P_3 \times P_n) + 1$ and $|S'| = \gamma(P_3 \times P_n)$. Hence $\gamma \gamma(P_3 \times P_n) = 2\gamma(P_3 \times P_n) + 1$.

\[\square\]

**Theorem 2.13** $\gamma \gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n)$.

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**Proof.**

$\{u_1, u_2, u_3, \ldots, u_n\}, \{v_1, v_2, v_3, \ldots, v_n\}, \{w_1, w_2, w_3, \ldots, w_n\}$ and $\{x_1, x_2, x_3, \ldots, x_n\}$ are the vertices of the $1^{st}$, $2^{nd}$, $3^{rd}$ and $4^{th}$ rows of the grid $P_4 \times P_n$ as shown in the figure. We have
\( \gamma(P_4 \times P_n) = \begin{cases} 
+1 & \text{if } n = 1, 2, 3, 5, 6, 9 \\
n & \text{otherwise} 
\end{cases} \)

**Case (1):** \( n = 3k \)
When \( n = 3 \), \( S = \{w_1, u_2, v_3, x_3\} \) and \( S' = \{v_1, x_2, u_3, w_3\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_3) = 2\gamma(P_4 \times P_3) = ii(P_4 \times P_3) \).
When \( n = 6 \), \( S = \{v_1, x_2, u_3, w_4, v_5, u_6, x_6\} \) and \( S' = \{w_1, u_2, x_3, v_4, u_5, x_5, w_6\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_6) = 2\gamma(P_4 \times P_6) = ii(P_4 \times P_6) \).
When \( n = 9 \), \( S = \{u_1, x_1, v_2, w_3, u_4, v_5, x_5, v_6, x_7, u_8, w_9\} \) and \( S' = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, w_8, u_9, x_9\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_9) = 2\gamma(P_4 \times P_9) = ii(P_4 \times P_9) \).
When \( n = 12 \), \( S = \{v_1, x_2, u_3, w_4, v_5, x_6, w_7, v_8, x_9, u_10, x_{11}, u_{12}\} \) and \( S' = \{w_1, u_2, x_3, v_4, w_5, x_6, w_7, u_8, x_9, w_{10}, x_{11}, u_{12}\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_{12}) = 2\gamma(P_4 \times P_{12}) = ii(P_4 \times P_{12}) \).

**Subcase (1.i):** \( n \) is odd, \( k = 5, 7, 9, \ldots \)
\( S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, w_{n-7}, u_{n-6}, v_{n-5}, x_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\} \)
and \( S' = \{v_1, x_2, u_3, v_4, u_5, x_6, w_7, v_8, x_9, \ldots, w_{n-8}, u_{n-7}, v_{n-6}, x_{n-5}, w_{n-4}, u_{n-3}, x_{n-2}, u_{n-1}, w_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Subcase (1.ii):** \( n \) is even, \( k = 6, 8, 10, \ldots \)
\( S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, w_{n-7}, u_{n-6}, v_{n-5}, x_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\} \)
and \( S' = \{w_1, u_2, x_3, v_4, u_5, x_6, w_7, u_8, x_9, \ldots, w_{n-8}, u_{n-7}, v_{n-6}, x_{n-5}, w_{n-4}, u_{n-3}, x_{n-2}, u_{n-1}, w_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Case (2):** \( n = 3k + 1 \)
When \( n = 1 \), \( S = \{u_1, w_1\} \) and \( S' = \{v_1, x_1\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Subcase (2.i):** \( n \) is odd, \( k = 2, 4, 6, \ldots \)
\( S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, x_{n-7}, v_{n-6}, x_{n-5}, u_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\} \)
and \( S' = \{w_1, u_2, x_3, v_4, u_5, x_6, w_7, u_8, x_9, \ldots, v_{n-7}, w_{n-6}, u_{n-5}, x_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Subcase (2.ii):** \( n \) is even, \( k = 1, 3, 5, \ldots \)
\( S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, \ldots, u_{n-7}, w_{n-6}, u_{n-5}, x_{n-4}, x_{n-2}, u_{n-1}, w_n\} \)
and \( S' = \{u_1, w_2, x_3, v_4, u_5, x_6, w_7, u_8, x_9, \ldots, u_{n-7}, w_{n-6}, u_{n-5}, x_{n-4}, x_{n-2}, u_{n-1}, w_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Case (3):** \( n = 3k + 2 \)
When \( n = 2 \), \( S = \{u_1, x_1, v_2\} \) and \( S' = \{v_1, u_2, x_2\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).
When \( n = 5 \), \( S = \{v_1, x_2, u_3, v_4, x_5\} \) and \( S' = \{w_1, u_2, x_3, v_4, u_5, w_5\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma\gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).
When \( n = 8 \), \( S = \{v_1, x_2, u_3, w_4, v_5, x_6, u_7, w_8\} \) and \( S' = \{w_1, u_2, x_3, v_4, w_5, u_6, x_7, v_8\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma \gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Subcase (3.i):** \( n \) is odd, \( k = 3, 5, 7, \ldots \)

\( S = \{v_1, x_2, u_3, w_4, u_5, x_6, v_7, x_8, \ldots, v_{n-7}, u_{n-6}, x_{n-5}, v_{n-4}, w_{n-3}, u_{n-2}, x_{n-1}, v_n\} \) and \( S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, \ldots, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma \gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Subcase (3.ii):** \( n \) is even, \( k = 4, 6, 8, \ldots \)

\( S = \{v_1, x_2, u_3, w_4, x_5, x_6, v_7, x_8, u_9, \ldots, x_{n-8}, v_{n-7}, x_{n-6}, u_{n-5}, w_{n-4}, v_{n-3}, x_{n-2}, u_{n-1}, w_n\} \) and \( S' = \{w_1, u_2, x_3, v_4, x_5, u_6, w_7, u_8, x_9, \ldots, u_{n-8}, w_{n-7}, x_{n-6}, u_{n-5}, v_{n-4}, u_{n-3}, x_{n-2}, u_{n-1}, v_n\} \) are two disjoint independent \( \gamma \)-sets. Hence \( \gamma \gamma(P_4 \times P_3) = 2\gamma(P_4 \times P_3) = ii(P_4 \times P_3) \). Thus from the above cases, \( \gamma \gamma(P_4 \times P_n) = 2\gamma(P_4 \times P_n) = ii(P_4 \times P_n) \).

**Theorem 2.14** For any 2 integers \( m \) and \( n \) with \( n \geq 5, m \geq 2 \) we can construct a tree \( T \) with \( \gamma \gamma(T) = n - m + 1 = ii(T) \).

**Proof.** Let \( m \) and \( n \) be 2 integers with \( n \geq 4 \) and \( m \geq 2 \). Let \( u \) be the root of \( T \) which is a pendant vertex and \( m \) be the degree of the support \( v \) of \( u \). Let the neighbors of \( v \) other than \( u \) be \( u_1, u_2, u_3, \ldots, u_{m-1} \). Each \( u_i \) can have any number of neighbors say \( u_{ij} \) other than \( v \) where \( 1 \leq i \leq m - 1 \) and \( j \geq 1 \) such that \( \sum\sum 1 = r \) and each \( u_{ijk}, k \geq 1 \) such that \( \sum\sum\sum 1 = s \). Then \( \sum\sum 1 + \sum\sum\sum 1 + m + 1 = n \). We now construct two disjoint dominating sets of \( T, S \) and \( S' \). Without loss of generality let \( u \in S \). Then as \( v \) is adjacent to \( u, u \notin S', v \in S' \). Then \( u_{ij}, 1 \leq i \leq m - 1 \) must be in \( S \). Since \( u_{ij} \) are adjacent to \( u_{ij}, u_{ijk} \in S' \). Thus the members of \( S \) are \( u \) and \( u_{ijk} \). Therefore \( S \) is a dominating set of \( T \). As \( v \in S' \), it dominates \( u \) and \( u_1, u_2, u_3, \ldots, u_{m-1} \). Hence \( u, u_1, u_2, u_3, \ldots, u_{m-1} \) \( \notin S' \). Let \( u_{ij} \in S' \). Then \( u_{ijk} \) dominates \( u_{ij} \). Thus the set \( S \) consisting of the vertices \( v \) and \( u_{ijk} \) is a dominating set of \( T \) disjoint from \( S \). There is no other dominating set disjoint with \( S \) or \( S' \) with minimum cardinality as any dominating set should contain either \( u \) or \( v \). Also \( S \) and \( S' \) are independent. Hence \( \gamma \gamma(T) = |S \cup S'| = n - m + 1 = ii(T) \).

**Theorem 2.15** For a \( n \)-star graph \( S_n, \gamma \gamma(S_n) = 2\gamma(S_n) \) and consequently \( S_n \) is a \( \gamma \gamma \)-minimum graph

**Proof.** We know that \( S_n \) is \((n - 1)\) regular. By Theorem 1.2 \( \gamma(S_n) = (n - 1)! \) and \( S_n \) is domatically full. Also every class of domatic partition of \( S_n \) is a \( \gamma \)-set of \( S_n \). Hence any two members of the domatic partition form a pair of disjoint dominating sets and so \( \gamma \gamma(S_n) = 2\gamma(S_n) \).
In [5], hypercubes are conjectured to be $\gamma\gamma$-minimum for all $n \geq 2$. We give a partial solution to this conjecture.

**Definition 2.16** [2] The $n$-cube $Q_n$ is a graph whose vertex set is the set of all $n$-dimensional boolean vectors, two vertices being joined if and only if they differ in exactly one co-ordinate.

We use the following notation. By $(0)$ we mean the boolean vector with all coordinates 0. If $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, we denote by $(i_1,i_2,\ldots,i_k)$ the $n$-tuple having 1 in the coordinates $i_1,i_2,\ldots,i_k$ and 0 elsewhere.

**Example 2.17** $\gamma\gamma(Q_1) = 2\gamma(Q_1)$ for $1 \leq n \leq 7$.

$\gamma\gamma(Q_1) = 2$ since $S = \{(1)\}$ and $S' = \{(2)\}$ are two disjoint dominating sets of $Q_1$. $\gamma\gamma(Q_2) = 4$ since $S = \{(1),(2)\}$ and $S' = \{(0),(1,2)\}$ are two disjoint dominating sets of $Q_2$. $\gamma\gamma(Q_3) = 4$ since $S = \{(1),(2),(3)\}$ and $S' = \{(1,2),(2,3)\}$ are two disjoint dominating sets of $Q_3$. $\gamma\gamma(Q_4) = 8$ since $S = \{(1),(2),(3),(4)\}$ and $S' = \{(1,2),(2,3),(3,4)\}$ are two disjoint dominating sets of $Q_4$. $\gamma\gamma(Q_5) = 14$ since $S = \{(0),(1,2),(1,3),(1,4,5),(2,3,4),(2,3,5),(2,3,4,5)\}$ and $S' = \{(1),(2),(3),(4,5),(1,2,3,4),(1,2,3,5),(1,2,3,4,5)\}$ are two disjoint dominating sets of $Q_5$. $\gamma\gamma(Q_6) = 24$ since $S = \{(0),(1,3),(2,3),(1,2,4),(1,3,4,5),(3,4,5),(3,4,6),(1,2,4,5),(1,2,4,6),(3,4,5,6),(1,2,3,4,5),(2,3,4,5),(2,3,5,6),(1,2,3,4,5),(2,3,4,5),(2,3,5,6),(1,2,3,4,5),(2,3,4,5),(2,3,5,6)\}$ and $S' = \{(1),(2),(3),(4,5),(1,2,3,4),(1,2,3,5),(1,2,3,4,5),(2,3,4,5),(2,3,5,6),(1,2,3,4,5),(2,3,4,5),(2,3,5,6)\}$ are two disjoint dominating sets of $Q_6$. $\gamma\gamma(Q_7) = 32$ since $S = \{(0),(1,2,7),(1,3,4),(1,5,6),(2,3,5),(2,2,6),(3,6,7),(4,5,7),(1,2,3,6),(1,2,4,5),(1,3,5,7),(1,4,6,7),(2,3,4,7),(2,5,6,7),(3,4,5,6),(1,2,3,4,5,6,7)\}$ and $S' = \{(1),(2,6),(3,4),(5,7),(2,3,7),(2,4,5),(3,5,6),(4,6,7),(1,2,3,5),(1,2,4,7),(1,3,6,7),(1,4,5,6),(1,2,3,6),(1,2,5,6,7),(1,3,4,5,7),(2,3,4,5,6,7)\}$ are two disjoint dominating sets of $Q_7$. Hence by [2], $\gamma\gamma(Q_n) = 2\gamma(Q_n)$ for $1 \leq n \leq 7$.

In this connection, we propose the following conjecture.

**Conjecture:** Hypercubes $Q_n$ are $\gamma\gamma$-minimum for $n \geq 8$.

**Theorem 2.18** Let $G$ be a graph without isolated vertices. Then $2 \leq \gamma\gamma(G) \leq p$. Lower bound is attained if and only if $G \cong K_n$ or $G$ has at least two vertices of full degree.

**Proof.** Obviously $2 \leq \gamma\gamma(G) \leq p$. Suppose $\gamma\gamma(G) = 2$. Then there exists two disjoint dominating sets $S$ and $S'$ such that both have cardinality one. This is possible if and only if $G \cong K_n$ or $G$ has at least two vertices of full degree. \qed

**Definition 2.19** The trestled graph of index $k$ denoted by $T_k(G)$ is a graph obtained from $G$ by adding $k$ copies of $K_2$ for each edge $uv$ of $G$ and joining $u$ and $v$ to the respective end vertices of each $K_2$. 
Theorem 2.20 If $G$ is a trestled graph of index $k$ of a cycle $C_n$, then $\gamma\gamma(G) = (k + 1)n$ where $k \in \mathbb{N}$.

Proof. The trestled graph of a cycle $C_n$ of index $k$ contains $n + 2kn = (2k + 1)n$ vertices. The set of $n$ vertices of the cycle $C_n$ say $S$ is a $\gamma$-set of $G$. The set of any one of the vertices of each of the newly added edge say $S'$ is another minimum dominating set of $G$ containing $nk$ vertices. Also $S \cap S' = \phi$. Hence $\gamma\gamma(G) = (k + 1)n$. \hfill $\Box$

Corollary 2.21 If $G \cong T_1(C_n)$ then $\gamma\gamma(G) = 2\gamma(G)$.

Corollary 2.22 If $G \cong T_k(C_n)$ then $ii(G) = 2kn$.

Theorem 2.23 If $G \cong T_k(P_n)$ then $\gamma\gamma(G) = n + k(n - 1)$.

Proof. The set of $n$ vertices of the path $P_n$ say $S$ is a $\gamma$-set of $T_k(P_n)$. Hence $|S| = n$. $P_n$ has $(n - 1)$ edges and corresponding to each edge there are $k$ edges. The set of one of the vertices of these $k(n - 1)$ edges say $S'$ form a dominating set of $T_k(P_n)$ and $S \cap S' = \phi$. Hence $|S'| = k(n - 1)$ and so $\gamma\gamma(T_k(P_n)) = n + k(n - 1)$. Thus $\gamma\gamma(T_k(P_n)) = n + k(n - 1)$. \hfill $\Box$

Theorem 2.24 If $G \cong T_m(K_{1,n})$ then $\gamma\gamma(T_m(K_{1,n})) = n + 1 + mn$.

Proof. The set of $(n + 1)$ vertices of the star say $S$ dominates $G$ and hence $|S| = n + 1$. $K_{1,n}$ has $n$ edges and corresponding to each edge there are $m$ edges. The set consisting of one vertex from each of the $nk$ edges $S'$ form an independent dominating set of $G$. Hence $|S'| = mn$. Also $S \cap S' = \phi$. Therefore $\gamma\gamma(T_m(K_{1,n})) = n + 1 + mn$. \hfill $\Box$

Corollary 2.25 $\gamma\gamma(T_1(K_{1,n})) = 2\gamma(T_1(K_{1,n}))$.

Proof. The $(n + 1)$ vertices of $K_{1,n}$ dominates $T_1(K_{1,n})$. Hence $|S| = n + 1$. The set consisting of one vertex from each of the newly added edge of $T_1(K_{1,n})$ together with the other vertex of the last edge say $S'$ form a dominating set of $T_1(K_{1,n})$ disjoint from $S$. Hence $|S'| = n + 1$ and so $\gamma\gamma(T_1(K_{1,n})) = 2(n + 1) = 2\gamma(T_1(K_{1,n}))$. \hfill $\Box$

Definition 2.26 The total graph $T(G)$ of a graph $G = (V, E)$ has vertices that correspond one to one with the elements of $V \cup E$. Two vertices in $T(G)$ are adjacent if and only if the corresponding elements are adjacent or incident in $G$.

Theorem 2.27 $\gamma\gamma(T(P_n)) = \begin{cases} 2\gamma(T(P_n)) + 1 & \text{if } n \equiv 3(\text{mod } 5) \\ 2\gamma(T(P_n)) & \text{otherwise} \end{cases}$
Hence \(\gamma(T)\) of vertex \(T\) is a minimal dominating set of \(T\). Thus \(|\gamma(T)| = 2\gamma(T)\).

Proof. Let \(\{v_1, v_2, v_3, \ldots, v_n\}\) be the vertex set of \(P_n\). As \(P_n\) contains \(n\) vertices, \(T(P_n)\) contains \((2n - 1)\) vertices, say \(v_1, e_1, v_2, e_2, v_3, \ldots, e_{n-1}, v_n\). It is easy to observe that \(\gamma(T(P_n)) = \lceil \frac{2n-1}{3} \rceil\).

Case(i): \(n \equiv 0 (mod\ 5)\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, v_{n-8}, e_{n-6}, v_{n-3}, e_{n-1}\}\] and

\[S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-9}, v_{n-6}, e_{n-4}, e_{n-2}, v_n\}\] are two disjoint \(\gamma\)-sets of \(T(P_n)\). Hence \(\gamma(T(P_n)) = 2\gamma(T(P_n))\).

Case(ii): \(n \equiv 1 (mod\ 5)\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, v_{n-9}, e_{n-6}, e_{n-4}, v_{n-1}\}\] and

\[S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-7}, v_{n-4}, e_{n-2}, v_n\}\] are two disjoint \(\gamma\)-sets of \(T(P_n)\). Hence \(\gamma(T(P_n)) = 2\gamma(T(P_n))\).

Case(iii): \(n \equiv 2 (mod\ 5)\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, e_{n-8}, v_{n-5}, e_{n-3}, v_{n-1}\}\] and

\[S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, v_{n-8}, e_{n-6}, v_{n-3}, e_{n-1}\}\] are two disjoint \(\gamma\)-sets of \(T(P_n)\). Hence \(\gamma(T(P_n)) = 2\gamma(T(P_n))\).

Case(iv): \(n \equiv 3 (mod\ 5)\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, e_{n-9}, v_{n-6}, e_{n-4}, v_{n-1}\}\] is the \(\gamma\)-set of \(T(P_n)\) and is unique. \(S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-7}, v_{n-4}, e_{n-2}, v_n\}\) is the minimal dominating set of \(T(P_n)\) disjoint from \(S\) and \(|S'| = |S| + 1\). Hence \(\gamma(T(p_n)) = 2\gamma(T(P_n)) + 1\).

Case(v): \(n \equiv 4 (mod\ 5)\)

\[S = \{v_2, e_4, v_7, e_9, v_{12}, \ldots, v_{n-7}, e_{n-5}, v_{n-2}, e_{n-1}\}\] and

\[S' = \{e_1, v_4, e_6, v_9, e_{11}, \ldots, e_{n-8}, v_{n-5}, e_{n-3}, v_n\}\] are two disjoint \(\gamma\)-sets of \(T(P_n)\). Hence \(\gamma(T(P_n)) = 2\gamma(T(P_n))\).

\[\square\]

Theorem 2.28 \(\gamma(T(C_n)) = 2 \lceil \frac{2n}{3} \rceil\).

Proof. \(T(C_n)\) contains \(2n\) vertices and each vertex is of degree 4 as each vertex of \(T(C_n)\) is incident with two vertices and two edges of \(C_n\). Since \(T(C_n)\) has \(2n\) vertices, we can construct two disjoint dominating set \(S\) and \(S'\) of \(T(C_n)\) in such a way that no two vertices of either \(S\) or \(S'\) dominates the same vertex. Thus \(|S| = |S'| = \lceil \frac{2n}{3} \rceil\) and therefore \(\gamma(T(C_n)) = 2 \lceil \frac{2n}{3} \rceil\).

\[\square\]

Theorem 2.29 \(\gamma(T(K_{1,n})) = n + 1\).

Proof. Let \(v\) be the center and \(v_1, v_2, v_3, \ldots, v_{n-1}\) be the pendant vertices of \(K_{1,n}\). Then \(T(K_{1,n})\) contains \((2n + 1)\) vertices. \(v\) is the universal vertex of \(T(K_{1,n})\). Hence \(S = \{v\}\) is the \(\gamma\)-set of \(T(K_{1,n})\) and \(S' = \{v_1, v_2, v_3, \ldots, v_{n-1}\}\) is a minimal dominating set of \(T(K_{1,n})\). Also \(S \cap S' = \emptyset\) and \(|S'| = n\). Any minimal dominating set other than \(S\) contains \(n\) vertices. Hence \(\gamma(T(K_{1,n})) = n + 1\).

\[\square\]
In [5], "when is $\gamma \gamma (G) + \gamma \gamma (\overline{G}) = n + 4?$" was posed as an open problem. We observe that if $G \cong C_4$ or a connected graph in which every vertex is a leaf or a stem then $\gamma \gamma (G) + \gamma \gamma (\overline{G}) = n + 4$.

**Example 2.30** The path $P_n$ when $n = 3k + 2$, $k \in \mathbb{N}$ has $\gamma \gamma (P_n) = \frac{2(n+1)}{3}$. 

$\gamma \gamma (P_n) = \gamma \gamma (P_{3k+2}) = 2\gamma (P_{3k+2}) = 2(k+1) = \frac{2n+2}{3} = \frac{2(n+1)}{3}$.

This example gives a partial answer to the question "for which class of trees $T$ is $\gamma \gamma (T) = \frac{2(n+1)}{3}$.

We also see that the cycle $C_n$ with $n = 3k + 2$, $k \in \mathbb{N}$ has $\gamma \gamma (C_n) = \frac{2(n+1)}{3}$.

For, $\gamma \gamma (C_n) = \gamma \gamma (C_{3k+2}) = 2\left\lceil \frac{3k+2}{3} \right\rceil = 2(k+1) = \frac{2(n+1)}{3}$.

**References**


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