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A Fixed Point Theorem on Quasi-Metric Spaces of Hyperbolic Type

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Abstract

In this paper, we first define the concept quasi- metric space of hyperbolic type and then prove the results [3] for quasi- metric space of hyperbolic type.

Keywords: quasi-metric, fixed point

1 Introduction

It is well-known that if (X, d) is a complete metric space and $T : X \to X$ is a self-mapping satisfying $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$, where $0 < \lambda < 1$, then T has a unique fixed point. Ciric [2] introduced and studied self-mappings on X satisfying

$$d(Tx, Ty) \leq \lambda \max\{d(x.y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where $0 < \lambda < 1$. Boyd and Wong [4] studied mappings which satisfy in the following from:

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is an upper semi-continuous from the right function,satisfying $\varphi(t) < t$ for all $t > 0$. Ciric [3] studied some fixed point theorems on metric space of hyperbolic type. In this paper, we prove these results for quasi- metric space of hyperbolic type.

2 Main Result

Let us recall that a quasi-metric on a nonempty set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$ the following statements hold.

i) $x = y$ if and only if $d(x, y) = d(y, x) = 0$.

ii) $d(x, y) \leq d(x, z) + d(z, y)$

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X. A quasi-metric space is called a quasi-metric space of hyperbolic type if it contains a family L of metric segments(isometric images of real line segments) such that

(a) each two points x, y in X are endpoints of exactly one member seg$[x, y]$ of L , and

(b) if $u, x, y \in X$ and if $z \in$ seg$[x, y]$ satisfies $d(x, z) = \lambda d(x, y)$ for $\lambda \in [0, 1]$, then

$$d(u, z) \leq (1 - \lambda)d(u, x) + \lambda d(u, y). \tag{1}$$

Define $\Phi = \{\varphi : \mathcal{R}^+ \to \mathcal{R}^+\}$, where $\mathcal{R}^+ = [0, +\infty)$ and each $\varphi \in \Phi$ satisfies the following conditions.

(a) φ is continuous from the right on $\mathcal{R}^+$.

(b) φ is non-decreasing.

(c) $\varphi(t) < t$ for each $t > 0$.

(d) $\lim_{n\to\infty} \sup(t - \varphi_j(t)) = \infty$.

Theorem 2.1 *Let (X, d) be a complete quasi-metric space of hyperbolic type, K be nonempty closed subset of X and $T : K \to X$ be a non-self mapping such that*

(i) *$T(\partial K) \subseteq K$.*

(ii) *$d(Tx, Ty) \leq \varphi_s(d_s)$ for all $x, y \in K$, where*

$$\varphi_s(d_s) = \max\{\varphi_1(d(x, y)), \varphi_2(d(x, Tx)), \varphi_3(d(y, Ty)), \varphi_4(d(x, Ty)), \varphi_5(d(y, Tx))\} \tag{2}$$

and $\varphi_j \in \Phi$ for $(j = 1, 2, 3, 4, 5)$.

Then T has a fixed point in K.

Proof. We prove theorem in three step.

Step 1. Choose $x_0 \in \partial K$. Then $Tx_0 \in K$. Set $x_1 = Tx_0$. If $Tx_1 \in K$, set $x_2 = Tx_1$. If $Tx_1 \notin K$, then by [1], there exists $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, Tx_1) = d(x_1, Tx_1),$$

that is, $x_2 \in$ seg$[x_1, Tx_1] \cap \partial K$. Continuing this process we can choose a sequence $\{x_n\}$ in K and a sequence $\{Tx_n\}$ in X such that if $Tx_{n-1} \in K$, then $x_n = Tx_{n-1}$ and if $Tx_{n-1} \notin K$, then $x_n \in \partial k$ and $x_n \in$ seg$[x_{n-1}, Tx_{n-1}]$. It is easy to see that if $Tx_{n-1} \notin K$, then $x_n \neq Tx_{n-1}$ and $x_{n-1} = Tx_{n-2}$. Hence in this case we have

$$x_n \in \text{ seg}[Tx_{n-2}, Tx_{n-1}] \cap \partial K \tag{3}$$

for all $n \geq 2$.

Step2. For each integer $n \geq 1$, we define

$$A_n = \{x_i\}_{i=0}^{n-1} \cup \{Tx_i\}_{i=0}^{n-1} \quad \alpha_n = \sup\{d(a,b), d(b,a) : a, b \in A_n\}.$$

Without loss of generality, we may assume that $\alpha_n > 0$ for all $n \in \mathcal{N}$. We consider the following cases.

Case 1. Let there exist $i, k \in \{0, 1, ..., n-1\}$ such that

$$\alpha_n = d(x_i, Tx_k).$$

If $x_i \neq x_0$, then $x_{i-1} \in K$ and so Tx_{i-1} is defined.

(a) Suppose that $Tx_{i-1} \in K$. Then $x_i = Tx_{i-1}$. Thus there exists $\varphi_{j_0}(t) \in \phi_5 = \{\varphi_1, ..., \varphi_5\}$ such that

$$\alpha_n = d(x_i, Tx_k) = d(Tx_k, Tx_{i-1}) \leq \varphi_{j_0}(d_{j_0}), \tag{4}$$

where

$$d_{j_0} \in \{d(x_k, x_{i-1}), d(Tx_{i-1}, x_{i-1}), d(Tx_k, x_k), d(Tx_k, Tx_{i-1}), d(x_{k-1}, Tx_k)\}.$$

Since $d_{j_0} \leq \alpha_n$ and $\phi_{j_0}(t)$ is non-decreasing, it follows that $\varphi_{j_0}(d_{j_0}) \leq \varphi_{j_0}(\alpha_n)$. Thus $\alpha_n \leq \varphi_{j_0}(\alpha_n)$, which is in contradiction with the hypothesis (c) for $\varphi_{j_0}(t)$. Therefore, $x_i = x_0$.

(b) Consider now the case $Tx_{i-1} \notin K$. Then $i \geq 2$, as $Tx_0 \in K$, and from (3),

$$x_i \in \text{seg}[Tx_{i-2}, Tx_{i-1}]. \tag{5}$$

It follows from (1) with $x = Tx_{i-2}, y = Tx_{i-1}, z = x_i$ and $u = Tx_k$, that

$$\begin{aligned} \alpha_n &= d(Tx_k, x_i) \leq (1-\lambda)d(Tx_k, Tx_{i-2}) + \lambda d(Tx_k, Tx_{i-1}) \\ &\leq \max\{d(Tx_k, Tx_{i-2}), d(Tx_k, Tx_{i-1})\}. \end{aligned}$$

Similar to (4) this relation leads to a contradiction. Therefore, if $\alpha_n = d(x_i, Tx_k)$, then $x_i = x_0$, i.e., $\alpha_n = d(x_0, Tx_k)$.

Case2. Let there exist $0 \leq i < j \leq n-1$ such that $\alpha_n = d(x_i, x_j)$. Note that in this case, $x_{j-1} \in k$, because $j > 0$.

(a) If $x_j = Tx_{j-1}$, then case 2(a) reduces to case 1(a).

(b) If $Tx_{j-1} \notin K$, then from (3) with $n = j (j \geq 2)$ we get $x_j \in \text{seg}[Tx_{j-2}, Tx_{j-1}]$. Since this relation is similar to (5), it is easy to see that case 2(b) reduces to case 2(a). The remaining case $\alpha = d(Tx_i, Tx_k)$ is impossible. Thus

$$\alpha_n = \max\{d(x_0, Tx_k\} : k \in \{0, 1, ..., n-1\}\} \tag{6}$$

for all $n \in \mathcal{N}$. By definition of the sequence $\{\alpha_n\}$, it is non-decreasing. Now if $\lim_{n\to\infty} \alpha_n = \infty$, then for each $\varphi_j(t) \in \phi_5$, there exists a positive number $\Delta_j = \Delta(\varphi_j)$ such that $\alpha_n - \varphi_j(\alpha_n) > d(x_0, Tx_0)$ for all $\alpha_n > \Delta_j$. Set $\Delta = \max\{\Delta_j : j \in \{1, 2, 3, 4, 5\}\}$. Then $\alpha_n - \varphi_j(\alpha_n) > d(x_0, Tx_0)$ for all $\varphi_j(t) \in \phi_5$ and $\alpha_n > \Delta$. Let n be any fixed integer such that $\alpha_n > \Delta$. From (2) and (6) we see that

$$\begin{aligned}
\alpha_n = d(x_0, Tx_{k(n)}) &\leq d(x_0, Tx_0) + d(Tx_0, Tx_{k(n)}) \\
&\leq d(x_0, Tx_0) + \varphi_{j(n)}(d_{j(n)}) \\
&\leq d(x_0, Tx_0) + \varphi_{j(n)}(\alpha_n) \\
&< \alpha_n - \varphi_{j(n)}(\alpha_n)(\alpha_n) + \varphi_{j(n)}(\alpha_n) \\
&= \alpha_n,
\end{aligned}$$

for some $0 < k(n) \leq n - 1$ and for some fixed $\varphi_{j(n)}(t) \in \varphi_5$, where

$$d_{j(n)} \in \{d(x_0, x_{k(n)}), d(x_0, Tx_0), d(x_{k(n)}, Tx_{k(n)}), d(x_0, Tx_{k(n)}), d(x_{k(n)}, Tx_0)\}.$$

This contradiction implies that $\lim_{n\to\infty} \alpha_n = \alpha < \infty$ and hence both sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded.

Step3. For each integer $n \geq 2$, we define

$$B_n = \{x_i\}_{i\geq n} \cup \{Tx_i\}_{i\geq n} \quad \text{and} \quad \beta_n = \sup\{d(a, b), d(b, a) : a, b \in B_n\}.$$

Then $\beta_n \leq \alpha$ and $\{\beta_n\}$ is a non-increasing sequence. Thus the sequence β_n converges to some β. We shall show that $\beta = 0$. By the proof of (6)

$$\beta_n = \sup\{d(x_n, Tx_k) : k \geq n\}.$$

Let $n \in \mathcal{N}$. Denote by $\varphi_{j_0}(t)$ one of the functions $\varphi_j(t) \in \phi_5$, such that

$$\varphi_{j_0}(\beta_{n-2}) = \max\{\varphi_j(\beta_{n-2}) : \varphi_j(t) \in \phi_5\}. \tag{7}$$

Fix $k \geq n$. If $x_n = Tx_{n-1}$, then from (2)

$$d(x_n, Tx_k) = d(Tx_{n-1}, Tx_k) \leq \varphi_{j(k)}(dj_{k(k)})$$

for some fixed$\varphi_{j(k)}(t) \in \phi_5$, where

$$d_{j(k)} \in \{d(x_{n-1}, x_k), d(x_{n-1}, Tx_{n-1}), d(x_k, Tx_k), d(x_{n-1}, Tx_k), d(x_k, Tx_{n-1})\}.$$

Clearly, $d_{j(k)} \leq \beta_{n-1} \leq \beta_{n-2}$. Since $\varphi_{j(k)}(t)$ is non-decreasing, it follows that

$$d(Tx_{n-1}, Tx_k) \leq \varphi_{j(k)}(\beta_{n-2}) \leq \varphi_{j_0}(\beta_{n-2}), \tag{8}$$

where $d(Tx_{n-1}, Tx_k) = d(x_n, Tx_k)$. Now if $x_n \neq Tx_{n-1}$, then $Tx_{n-1} \notin K$ and so $x_n \in \text{seg}[Tx_{n-2}, Tx_{n-1}]$. Since X is of hyperbolic type, from (1) with $x = Tx_{n-2}, y = Tx_{n-1}, z = x_n$ and $u = Tx_k$, we get

$$d(Tx_k, x_n) \leq (1-\lambda)d(Tx_k, Tx_{n-2}) + \lambda d(Tx_k, Tx_{n-1}). \tag{9}$$

Similarly to (8) we have

$$d(Tx_k, Tx_{n-1}), d(Tx_k, Tx_{n-2}) \leq \varphi_{j_0}(\beta_{n-2}).$$

Thus, from (8) and (9), in any case we have $d(Tx_k, x_n) \leq \varphi_{j_0}(\beta_{n-2})$. Therefore,

$$\beta_n \leq \varphi_{j_0}(\beta_{n-2}). \tag{10}$$

Suppose that $\beta > 0$. Choose $\varphi_j(t) \in \phi_5$ such that

$$\varphi_{j(\beta)}(\beta) = \max\{\varphi_{j(\beta)} : \varphi_j(t) \in \phi_5\}.$$

Then from property (c) of ϕ we have $\eta = \beta - \varphi_{j((\beta)}(\beta) > 0$. It follows from properties (a) and (b) of Φ that there exists $\delta = \delta(\eta)$ such that

$$|\varphi_j(\beta_n) - \varphi_j(\beta)| < \eta$$

for all $\varphi_j(t) \in \phi_5$ and $\beta_n < \beta + \delta$. Hence

$$\varphi_j(\beta_n) < \varphi_j(\beta) + \eta \leq \varphi_{j(\beta)}(\beta) + \eta = \beta \tag{11}$$

for all $\beta_n < \beta + \delta$ and each $\varphi_j(t) \in \phi_5$. Let n be fixed integer such that $\beta_{n-2} < \beta + \delta$. Then from (11) we have $\varphi_{j_0}(\beta_{n-2}) < \beta$, where $\varphi_{j_0}(t) \in \phi_5$ is chosen so that (7) holds. Since $\beta \leq \beta_n$, from (10) we have $\beta \leq \varphi_{j_0}(\beta_{n-2}) < \beta$, a contradiction. Thus $\beta = 0$ and so $\lim_{n\to\infty} \beta_n = \beta = 0$. From this and the definition of β_n we conclude that both $\{x_n\}$ and $\{Tx_n\}$ are cauchy sequences. Since X is complete and K is closed, there is some $z \in K$ such that

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n. \tag{12}$$

We show that $Tz = z$. Suppose, to the contrary that

$$d(z, Tz) > 0 \quad \text{or} \quad d(Tz, z) > 0.$$

First, let $d(z, Tz) > 0$. Fix $\varphi_{j(z)}(t) \in \phi_5$ such that

$$\varphi_{j(z)}(d(z, Tz)) = \max\{\varphi_j(d(z, Tz)) : \varphi_j(t) \in \phi_5\}. \tag{13}$$

Then by property (c) of φ,

$$\eta = (d(z, Tz) - \varphi_{j(z)}(d(z, Tz)))/2 > 0. \tag{14}$$

From (2), for each fixed n , we get

$$d(Tx_n, Tz) \leq \varphi_{j(n)}(d_{j(n)}) \tag{15}$$

for some fixed $\varphi_{j(n)}(t) \in \phi_5$, where

$$d_{j(n)} \in \{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(x_n, Tz), d(z, Tx_n)\}. \tag{16}$$

Since each of the five functions $\varphi_j(t)$ is non-decreasing and continuous from the right at a point $t = d(z, Tz)$, there is an $\delta = \delta(\eta) > 0$ such that

$$\varphi_{j(n)}(t) < \varphi_{j(n)}(d(z, Tz)) + \eta \tag{17}$$

for all $t < d(z, Tz) + \delta$ and $n \in \mathcal{N}$. From (12) and (16) follows that there is $n_0 \in \mathcal{N}$ such that $d_{j(n)} < d(z, Tz) + \delta$ for all $n > n_0$. Thus from (15) and (17), with $t = d_{j(n)}$, and (13) we get

$$\begin{aligned} d(Tx_n, Tz) &< \varphi_{j(n)}(d(z, Tz)) + \eta \\ &< \varphi_{j(z)}(d(z, Tz)) + \eta. \end{aligned}$$

By (12) and (14),

$$\begin{aligned} d(z, Tz) &\leq \varphi_{j(z)}(d(z, Tz)) + \eta \\ &= (d(z, Tz) + \varphi_{j(z)}(d(z, Tz))/2 \\ &< d(z, Tz), \end{aligned}$$

a contradiction. Thus, $d(z, Tz) = 0$. Similarly, $d(Tz, z) = 0$. Hence $Tz = z$.

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