∞-Tuples of Operators and Hereditarily

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Abstract

In this paper, we introduce for an ∞-tuple of operators on a Banach space and some conditions to an ∞-tuple to satisfying the HypercyclicityCriterion.

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1 Introduction

Let $\mathcal{X}$ be an infinite dimensional Banach space and $T_1, T_2, \ldots$ are commutative bounded linear operators on $\mathcal{X}$. By an ∞-tuple we mean the ∞-component $\mathcal{T} = (T_1, T_2, \ldots)$. For the ∞-tuple $\mathcal{T} = (T_1, T_2, \ldots)$ the set

$$\mathcal{F} = \{T_1^{k_1}T_2^{k_2} \ldots : k_i \geq 0, i = 1, 2, \ldots, n\}$$
is the semigroup generated by $T$. For $x \in \mathcal{X}$, take

$$\text{Orb}(T, x) = \{ Sx : S \in \mathcal{F} \}.$$ 

In other hand

$$\text{Orb}(T, x) = \{ T_1^{k_1}T_2^{k_2}\ldots(x) : k_i \geq 0, i = 1, 2, \ldots \}.$$ 

The set $\text{Orb}(T, x)$ is called orbit of vector $x$ under $T$ and $\infty$-Tuple $T = (T_1, T_2, \ldots)$ is called hypercyclic $\infty$-tuple, if there is a vector $x \in \mathcal{X}$ such that, the set $\text{Orb}(T, x)$ is dense in $\mathcal{X}$, that is

$$\text{Orb}(T, x) = \{ T_1^{k_1}T_2^{k_2}\ldots(x) : k_i \geq 0, i = 1, 2, \ldots \} = \mathcal{X}.$$ 

In this case, the vector $x$ is called a hypercyclic vector for the $\infty$-tuple $T$.

2 Preliminary Notes / Materials and Methods

**Definition 2.1** Let $\mathcal{V}$ be a topological vector space (TVS) and $T_1, T_2, \ldots$ are bounded linear mapping on $\mathcal{V}$, and $T = (T_1, T_2, \ldots)$ be an $\infty$-tuple of operators. The $\infty$-tuple $T$ is called weakly mixing if

$$T \times T \times \ldots : \mathcal{X} \times \mathcal{X} \times \ldots \to \mathcal{X} \times \mathcal{X} \times \ldots$$

is topologically transitive.

**Definition 2.2** Let

$$\{ m_{(k, 1)} \}_{k=1}^{\infty}, \{ m_{(k, 2)} \}_{k=1}^{\infty}, \ldots$$

be increasing sequences of non-negative integers. The $\infty$-tuple $T = (T_1, T_2, \ldots)$ is called hereditarily hypercyclic with respect to

$$\{ m_{j, 1} \}_{j=1}^{\infty}, \{ m_{j, 2} \}_{j=1}^{\infty}, \ldots$$

if for all subsequences

$$\{ m'_{j, 1} \}_{j=1}^{\infty}, \{ m'_{j, 2} \}_{j=1}^{\infty}, \ldots$$

of

$$\{ m_{j, 1} \}_{j=1}^{\infty}, \{ m_{j, 2} \}_{j=1}^{\infty}, \ldots$$

respectively, the sequence

$$\{ T_1^{m'_{(k, 1)}}T_2^{m'_{(k, 2)}}\ldots \}$$

is hypercyclic. In the other hand, there exists a vector $x$ in $\mathcal{X}$ such that

$$\{ T_1^{m_{(k, 1)}}T_2^{m_{(k, 2)}}\ldots(x) \} = \mathcal{X}.$$
**Note 2.3** Note that, if $X$ be an finite dimensional Banach space, then there are no hypercyclic operator on $X'$, also there are no $\infty$-tuple or $n$-tuple on $X$.

All of operators in this paper are commutative bounded linear operators on a Banach space. Also, note that by $\{j, i\}$ or $(j, i)$ we mean a number, that was showed by this mark and related with this indexes, not a pair of numbers. Readers can see [1 − 10] for some information.

### 3 Results and Discussion

These are the main results of the paper.

If the $\infty$-tuple of continuous linear mappings satisfying hypothesis of bellow theorem then we say that is satisfying the Hypercyclicity Criterion.

**Theorem 3.1 (The Hypercyclicity Criterion for $\infty$-Tuples)** Let $X$ be a separable Banach space and $T = (T_1, T_2, ...)$ is an $\infty$-tuple of continuous linear mappings on $X$. If there exist two dense subsets $Y$ and $Z$ in $X$, and strictly increasing sequences $\{m_{j,1}\}_{j=1}^\infty, \{m_{j,2}\}_{j=1}^\infty, ...$ such that:

1. $T_1^{m_{j,1}}T_2^{m_{j,2}}... \to 0$ on $Y$ as $j \to \infty$, 
2. There exist functions $\{S_j : Z \to X\}$ such that for every $z \in Z, S_jz \to 0$, and $T_1^{m_{j,1}}T_2^{m_{j,2}}...S_jz \to z$, on $Z$ as $j \to \infty$, then $T$ is a hypercyclic $\infty$-tuple.

Now, the main theorem of this paper is the bellow theorem.

**Theorem 3.2** An $\infty$-tuple $T = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to increasing sequences of non-negative integers

$$\{m_{j,1}\}_{j=1}^\infty, \{m_{j,2}\}_{j=1}^\infty, ...$$

if and only if for all given any two open sets $U, V$, there exist some positive integers $M_1, M_2, ...$ such that

$$T_1^{m_{k,1}}T_2^{m_{k,2}}...(U) \cap V \neq \phi$$

for

$$\forall m_{k,1} > M_1, \forall m_{k,2} > M_2, ...$$

**Proof.** Let $T = (T_1, T_2, ...)$ be hereditarily hypercyclic $\infty$-tuple with respect to increasing sequences of non-negative integers

$$\{m_{j,1}\}_{j=1}^\infty, \{m_{j,2}\}_{j=1}^\infty, ...$$
and suppose that there exist some open sets $\mathcal{U}$, $\mathcal{V}$ such that

$$T_1^{m_{k,1}}T_2^{m_{k,2}}...(\mathcal{U}) \cap \mathcal{V} = \phi$$

for some subsequence

$$\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$$

of

$$\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$$

respectively. Since the $\infty$-tuple $\mathcal{T} = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to

$$\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$$

and $\mathcal{U}, \mathcal{V}$ are open sets in $\mathcal{X}$, satisfying

$$T_1^{m_{k,1}}T_2^{m_{k,2}}...(\mathcal{U}) \cap \mathcal{V} \neq \phi$$

for any

$$m_{(k,j)} > M_j, j = 1, 2, ...$$

So there exist $(i, j)$, large enough for $j = 1, 2, ...$ such that $m_{(k,i)} > M_j$ for $j = 1, 2, ...$ and

$$T_1^{m_{(k,1)}}T_2^{m_{(k,2)}}...(\mathcal{U}) \cap \mathcal{V} \neq \phi.$$

This implies that

$$\{T_1^{m_{(k,1)}}T_2^{m_{(k,2)}}\}$$

is hypercyclic, so the $\infty$-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is indeed hereditarily hypercyclic with respect to the sequences

$$\{m_{(k,1)}\}_{k=1}^{\infty}, \{m_{(k,2)}\}_{k=1}^{\infty}, ....$$

By this the proof is complete.
Theorem 3.3 An $\infty$-tuple $\mathcal{T} = (T_1, T_2, \ldots)$ is hereditarily hypercyclic with respect to increasing sequences of non-negative integers $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots$, if and only if for all given any two open sets $\mathcal{U}, \mathcal{V}$, there exist some positive integers $M_1, M_2, \ldots$ such that $T_1^{m_{k,1}} T_2^{m_{k,2}} \ldots (\mathcal{U}) \cap \mathcal{V} \neq \phi$ for $\forall m_{k,1} > M_1, \forall m_{k,2} > M_2, \ldots$.

Proof. Let $\mathcal{T} = (T_1, T_2, \ldots)$ be hereditarily hypercyclic $\infty$-tuple with respect to increasing sequences of non-negative integers $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots$, and suppose that there exist some open sets $\mathcal{U}, \mathcal{V}$ such that $T_1^{m_{k,1}} T_2^{m_{k,2}} \ldots (\mathcal{U}) \cap \mathcal{V} = \phi$ for some subsequence $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, \ldots$ of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots$ respectively. Since the $\infty$-tuple $\mathcal{T} = (T_1, T_2, \ldots)$ is hereditarily hypercyclic with respect to $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots$, thus $\{T_1^{m'_{k,1}} T_2^{m'_{k,2}} \ldots \}$ is hypercyclic, and so we get a contradiction.

Conversely, suppose that $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, \ldots$ are arbitrary subsequences of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, \ldots$ respectively, and $\mathcal{U}, \mathcal{V}$ are open sets in $\mathcal{X}$, satisfying

$$T_1^{m_{k,1}} T_2^{m_{k,2}} \ldots (\mathcal{U}) \cap \mathcal{V} \neq \phi$$

for any $m_{(k,j)} > M_j, j = 1, 2, \ldots$. So there exist $(i, j)$, large enough for $j = 1, 2, \ldots$ such that $m_{(k,i)} > M_j$ for $j = 1, 2, \ldots$ and

$$T_1^{m_{(k,1)}} T_2^{m_{(k,2)}} \ldots (\mathcal{U}) \cap \mathcal{V} \neq \phi.$$

This implies that $\{T_1^{m_{(k,1)}} T_2^{m_{(k,2)}} \ldots \}$ is hypercyclic, so the $\infty$-tuple $\mathcal{T} = (T_1, T_2, \ldots)$ is indeed hereditarily hypercyclic with respect to the sequences $\{m_{(k,1)}\}_{k=1}^{\infty}, \{m_{(k,2)}\}_{k=1}^{\infty}, \ldots$.

By this the proof is complete.

References


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