Kaluza-Klein Cosmology with Variable G and $\Lambda$ Term

R. K. Dubey
Department of Mathematics, Govt. Science P.G. College
Rewa, Madhya Pradesh, India
rkdubey2004@yahoo.co.in

Abhijeet Mitra
Department of Mathematics, Govt. P G College
Satna, Madhya Pradesh, India
abhijeetmitra2@rediffmail.com

Bijendra Kumar Singh
Department of Mathematics, Govt. P. G. College
Satna, Madhya Pradesh, India
singh.bijendrakumar@gmail.com

Abstract. In this paper we have discussed kaluza-klein higher dimensional cosmological model with variable cosmological constant ($\Lambda$) and variable gravitational constant ($G$) under some suitable assumptions. We obtain the exact solutions for the field equations and discussed two models. An expanding universe is found by using a relation between scalar potential and an equation of state.

Keywords: Cosmology, Kaluza-Klein higher dimension, variable gravitational coupling ($G$) and Cosmological Constant term ($\Lambda$).

1. INTRODUCTION

In 1926 the Swedish physicist Oskar Klein came up with some major improvements to Kaluza’s theory, at which time it became universally known as Kaluza-Klein theory[5,8]. The idea of Kaluza-Klein has been worked by a large number of people, who have found models for various phenomena in particle physics and cosmology using five dimension or more dimensions [2-3]. Kaluza-Klein achievements have shown that five dimensional general relativity contains both Einstein’s four-dimensional theory of gravity and Maxwell’s theory of electromagnetism. Chatterjee and Banerjee [6](1993)and Banerjee et al[1](1995) have studied Kaluza-Klein inhomogeneous cosmological model with and
without cosmological constants respectively. So far there have been many cosmological solutions dealing with higher dimensional model containing a variety of matter field. However, there is a few work in literature where variable G and $\Lambda$ have considered in higher dimension[9].

A number of authors have said in favors of the dependence first expressed by Bertolami[4]and later on by several authors in different context. Recently Ray and Mukhopadhyay [7] have solved Einstein’s equation for specific dynamical models of the cosmological term in form $\Lambda = \left( \frac{R}{R} \right)$, $\Lambda = \left( \frac{R}{R} \right)$ and $\Lambda \sim \rho$ shown that the model are equivalent in the framework of flat RW space time. In this paper the implication of cosmological model with cosmological term of two different forms $\Lambda = \alpha \left( \frac{R}{R} \right)^2$ and $\Lambda = \beta \frac{R}{R}$ where $\alpha$ and $\beta$ are free parameters, are analyzed within the framework of higher dimensional space time.

2. THE FIELD EQUATION AND ITS SOLUTION

Let us consider, 5-dimension Kulaza-klein (kk) type Robertson-walker metric

$$ds^2 = dt^2 - R^2(t) \left[ \frac{dy^2}{(1-k\gamma^2)} + \gamma^2 (d\phi^2 + \sin^2 \phi d\varphi^2) + (1-k\gamma^2) d\gamma^2 \right]$$

(1)

where $R(t)$ is the scale factor and $k=0,-1$ or $+1$ the curvature parameter for flat, open and closed universe respectively .We assume the universe to be filled with matter and the distribution of matter is represented by energy-momentum tensor of a perfect fluid.

$$T_{ij} = (p+\rho)u_i u_j - pg_{ij}$$

(2)

Where $\rho$ is the energy density of the cosmic matter and $P$ is its pressure.

The Einstein field equations with time varying cosmological gravitational constant are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij} + \Lambda(t) g_{ij}$$

(3)

from (2.2) we get

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G [(p+\rho)u_i u_j - \rho g_{ij}] + \Lambda(t) g_{ij}$$

(4)

the above equation(2.4) for the metric(2.1) yield two independent equations.
Kaluza-Klein cosmology with variable $G$ and $\Lambda$ term

\[
6 \left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} = 8 \pi G \rho + \Lambda \tag{5}
\]

\[
3 \frac{\ddot{R}}{R} + 3 \left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} = -8 \pi G \rho + \Lambda \tag{6}
\]

Here equation (5) is time-time component and equation (6) is space-space component of the field equation (4). The over-dot denotes derivative w.r.t. time. Solving equation (5) and (6) we get the continuity equation.

\[
\left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} = \frac{8 \pi G \rho + \Lambda}{6}
\]

We obtain \( \ddot{R} = \frac{R^2}{12} \left( 8 \pi G \dot{\rho} + 8 \pi G \dot{\rho} + \dot{\Lambda} \right) + \frac{\dot{R}^2}{RR} + \frac{K \dot{R}}{RR} \) \tag{7}

Substituting the value of $\ddot{R}$ in equation (2.6) we have the continuity equation.

\[
\dot{\rho} + \frac{G \rho}{G} + \frac{\dot{\Lambda}}{8 \pi G} + 4(p+\rho) \frac{\dot{R}}{R} = 0 \tag{8}
\]

from equation (8) we observe that energy density is not conserved for matter field due to varying nature of scalars $G$ and $\Lambda$. Since the principle of equivalence requires only $g_{\mu \nu}$ and $\Lambda$ and $G$ should not involve. So in this case law of conservation of energy momentum holds and its shows from equation (8)

\[
\dot{\rho} + 4(p+\rho) \frac{\dot{R}}{R} = 0 \quad \text{using equation (9), equation (8)}
\]

\[
\frac{G \rho}{G} = -\frac{\dot{\Lambda}}{8 \pi G} \Rightarrow \dot{G} = -\frac{\dot{\Lambda}}{8 \pi \rho} \tag{10}
\]

using equation of state

\[
p = (\gamma-1) \rho \tag{11}
\]

where state parameter $\gamma$ can take constant value +1, 4/3, 0.2 respectively for dust radiation, vacuum fluid and stiff fluid. Using (11) in equation (9) we obtain
\[ \dot{\rho} + 4(\rho + (\gamma - 1)\rho) \frac{\dot{R}}{R} = 0 \quad \dot{\rho} + 4\gamma \rho \frac{\dot{R}}{R} = 0 \quad \frac{\dot{\rho}}{\rho} = -4\gamma \frac{\dot{R}}{R} \]  \hspace{1cm} (12)

Integrating we obtain

\[ \log \rho = -4\gamma \log R + \log b_2 \]  \hspace{1cm} (13)
\[ \therefore \rho = b_2 R^{-4\gamma} \]

Where \( b_2 = \rho_0 R_0^{-4\gamma} \) and suffix 0 represents the present value of parameters.

From equation (12) we get \[ \frac{\dot{R}}{R} = -\frac{1}{4\gamma} \frac{\dot{\rho}}{\rho} \]  \hspace{1cm} (14)

Substituting this value of \( \frac{\dot{R}}{R} \) in equation (5) we get

\[ \frac{\ddot{\rho}}{\rho^2} = 16\gamma^2 \left[ \frac{16\pi G}{12} + \frac{2\Lambda}{12\rho} - \frac{k}{R^2\rho} \right] \]  \hspace{1cm} (15)

Again differentiating equation (15) and using equation (10)

\[ \frac{2\ddot{\rho}}{\rho^3} = 3\left(\frac{\dot{\rho}}{\rho}\right)^2 = 16\gamma^2 \left[ -\frac{\dot{\rho}\Lambda}{6\rho^2} - \frac{2k}{4\gamma \rho^3 R^2} - \frac{k\dot{\rho}}{R^2\rho^2} \right] \]

Above equation is obtained using equations (10 & 12) so we have

\[ \frac{2\ddot{\rho}}{\rho} - 3\left(\frac{\dot{\rho}}{\rho}\right)^2 = 4\gamma^2 \left[ \frac{2k - 4\gamma k}{3} \right] \]  \hspace{1cm} (16)

Since from equation (14) \[ \frac{\dot{R}}{R} = -\frac{1}{4\gamma} \frac{\dot{\rho}}{\rho} \]

\[ \therefore H = -\frac{1}{4\gamma} \frac{\dot{\rho}}{\rho} \quad \text{As} \quad \frac{\dot{R}}{R} = \frac{\dot{R}}{R} \quad \text{is the Hubble parameter.} \]

\[ 2H = \frac{2}{4\gamma} \left(\frac{\ddot{\rho}}{\rho}\right) = -\frac{2}{4\gamma} \left(\frac{\dot{\rho}}{\rho}\right)^2 \]  \hspace{1cm} (17)

Substituting equation (17) in equation (16) we get
\[ \dot{H} + 2\gamma H^2 + \frac{(1-2\gamma)k}{R^2} - \frac{\Lambda \gamma}{3} = 0 \]

\[ 3 \frac{\ddot{R}}{R} + 3(2\gamma - 1) \left( \frac{\dot{R}}{R} \right)^2 + 3(2\gamma - 1) \frac{k}{R^2} = \Lambda \gamma \]  

(18)

From the above equation (18) it is clear that \( \Lambda \) depends on \( \frac{\dot{R}}{R}, \frac{\dot{R}}{R}, \frac{1}{R^2}, \rho \).

2.3 DIFFERENT COSMOLOGICAL MODELS

2.3.1 Model for \( \Lambda \sim \left( \frac{\dot{R}}{R} \right)^2 \)

Let us consider \( \Lambda = \alpha H^2 \) where \( H = \frac{\dot{R}}{R} \)

\[ 3 \dot{H} + 6\gamma H^2 + \frac{3(1-2\gamma)k}{R^2} = \alpha H^2 \gamma \]  

(19)

We consider the case when \( k=0 \) i.e. for flat universe then from equation (19)

\[ 3 \dot{H} = \gamma (\alpha - 6) H^2 \]

\[ R = t^{\gamma/(6-\alpha)} \]

\[ \Lambda = \frac{9\alpha}{\gamma^2(6-\alpha)^2} \frac{1}{t^2} \]  

thus \( \Lambda \sim t^{-2} \) which is in accordance with what Bertolami (3)said

From equation (13)

\[ \rho = b_2 \left( t^{\gamma/(6-\alpha)} \right)^{-4\gamma} \]

From equation (10)

\[ G = \frac{9\alpha}{8\pi b_2 \gamma^2(6-\alpha)} \frac{t^{2\gamma/(6-\alpha)}}{\alpha} \]
\[ G \sim t^{2\alpha/(6-a)} \]

### 2.3.2 Model for \( \Lambda \sim \frac{\ddot{R}}{R} \)

Let us consider \( \Lambda = \beta \frac{\ddot{R}}{R} \) then from equation (18) we have for \( k = 0 \)

\[ (3-\beta \gamma) \frac{\ddot{R}}{R} + 3(2\gamma-1)R^2 = 0 \]

\[ \therefore R = \frac{3(1-2\gamma)}{(3-\beta \gamma)} t^2 \]

\[ \Lambda = \beta \frac{6(1-2\gamma)}{(3-\beta \gamma)} \frac{1}{t^2} = \frac{2\beta}{t^2} \therefore \Lambda \sim t^{-2} \]

\[ \rho = b_2 R^{-4\gamma} = \frac{b_2}{2} \left( \frac{3(1-2\gamma)}{(3-\beta \gamma)} \right)^{-4\gamma} \]

\[ G = \beta \frac{t^{8\gamma}}{2\pi b_2 \left( \frac{3(1-2\gamma)}{(3-\beta \gamma)} \right)^{-4\gamma}} \]

### 2.4 CONCLUSION

In this chapter we have discussed Kaluza-Klein type Robertson-walker cosmological models by considering three different forms of variable \( \Lambda \) in respect of variable \( G \) we have obtained the exact solution of the field equations.

Since \( \frac{\ddot{R}}{R} = \dot{H} + H^2 \) then \( \Lambda \sim \frac{\ddot{R}}{R} \) models can be looked as a combination of the two models i.e. \( \Lambda \sim \dot{H}, \Lambda \sim H^2 \) we observe that \( \Lambda \sim \frac{\ddot{R}}{R} \) and \( \Lambda \sim \left( \frac{\ddot{R}}{R} \right) \) models become identical when \( \dot{H} = 0 \) i.e. when \( H \) is constant. In this case we get exponential expression and hence an inflationary scenario. Thus the concept of inflation is inherent in
phenomenological model \( \Lambda \sim \frac{\dot{R}}{R} \). Moreover \( \Lambda \sim \left( \frac{\dot{R}}{R} \right)^2 \) and \( \Lambda = \frac{\ddot{R}}{R} \) models cannot exit as separate entity during inflations. By using equation of state of the form \( p = (\gamma - 1) \rho \), we have found exact solutions of the field equations for these different cases \( \Lambda = \alpha H^2, \Lambda = \beta \frac{\ddot{R}}{R} \). By selecting a simple power law expression of \( t \) for the equation of state parameter \( \gamma \) such as \( \gamma \sim t \), equivalence of models \( \Lambda \sim \left( \frac{\dot{R}}{R} \right)^2, \Lambda \sim \left( \frac{\ddot{R}}{R} \right) \) can be established in the framework of Kaluza-Klein theory.

REFERENCES


4) O. Bertolami, Nuovo Cimento 93, 36 (1986); Fortschr. Phys. 34, 829(1986)

5) O. Klein (1926). Zeils. Phys 37,895

6) S. Chatterjee, and A. Banerjee (1993), Kaluza-Klein type of inhomogeneous cosmological models, *Classical and Quantum Gravitation*, Vol. 10, L1


Received: June, 2012