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Intuitionistic Fuzzy H-Ideals of BCI-Algebras with Interval Valued Membership & Non Membership Functions

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Abstract. The purpose of this paper is to define the notion of an interval valued intuitionistic fuzzy H-ideal (briefly, an i-v IF H-ideal) of a BCI- algebra. Necessary and sufficient conditions for an i-v intuitionistic fuzzy H-ideal are stated. Cartesian product of i-v intuitionistic fuzzy ideals are discussed.

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1. Introduction

The notion of BCK-algebras was proposed by Imai and Iseki in 1966. In the same year, Iseki [6] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [10]. In [9], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set(i.e., a fuzzy set with an interval valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. In [9], Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. In [4], Biswas defined interval-valued fuzzy subgroups (i.e., i-v fuzzy subgroups) of Rosenfeld's nature, and investigated some elementary properties. The idea of "intuitionistic fuzzy set" was first published by Atanassov[1],[2] as a generalization of the notion of fuzzy sets. After that many researchers considered the fuzzification of ideals and subalgebras in BCK(BCI)-algebras. In this paper, using the notion of interval-valued fuzzy set, we introduce the concept of an interval-valued intuitionistic fuzzy BCI-algebra (briefly, i-v IF subalgebra) of a BCI-algebra, and study some of their properties. Using an i-v level set of an i-v intuitionistic fuzzy set, we state a characterization of an intuitionistic fuzzy H-ideal of BCI-algebras. We prove that every intuitionistic fuzzy H-ideal of a BCI-algebra X can be realized as an i-v level H-ideal of an i-v intuitionistic fuzzy H-ideal of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v intuitionistic fuzzy H-ideal become i-v intuitionstic fuzzy H-ideal.

2. Preliminaries

Let us recall that an algebra $(X, *, 0)$ of type (2, 0) is called a BCI-algebra if it satisfies the following conditions:

1. $((x*y)*(x*z))*(z*y)=0,$
2. $(x*(x*y))*y=0,$
3. $x*x=0,$
4. $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y,z\in X$.

In a BCI-algebra, we can define a partial ordering "$\leq$" by $x\leq y$ if and only if $x*y=0$. In a BCI-algebra X, the set $M=\{x\in X/0*x=0\}$ is a subalgebra and is called the BCK-part of X. A BCI-algebra X is called proper if $X-M\neq\phi$. Otherwise it is improper. Moreover, in a BCI-algebra the following conditions hold:

5. $(x*y)*z=(x*z)*y,$
6. $x*0=0,$
7. $x\leq y$ imply $x*z\leq y*z$ and $z*y\leq z*x,$
8. $0*(x*y)=(0*x)*(0*y),$
9. $0*(0*(x*y))=0*(y*x),$
10. $(x*z)*(y*z)\leq x*y.$

An intuitionistic fuzzy set A in a non-empty set X is an object having the form

$$A=\{\langle x,\mu_A(x),\upsilon_A(x)\rangle \setminus x\in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\upsilon_A : X \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of non membership* (namely $\upsilon_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \le \mu_A(x) + \upsilon_A(x) \le 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for Example two journals:

1. Fuzzy Sets and 2. Notes on Intuitionistic Fuzzy Sets) and have many interesting applications not only in mathematics (See Chapter 5 in the book [2]).

For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \upsilon_A \rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \upsilon_A(x) \rangle \setminus x \in X\}$.

Definition 2.1. A non empty subset I of X is called an ideal of X if it satisfies:

1. $0 \in I$,
2. $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.2. A fuzzy subset μ of a BCI-algebra X is called an fuzzy ideal of X if it satisfies:

1. $\mu(0) \ge \mu(x)$,
2. $\mu(x) \ge \min\{\mu(x * y), \mu(y)\}$, for all $x, y \in X$.

Definition 2.3. A non empty subset I of X is called an H-ideal of X if it satisfies:

1. $0 \in I$,
2. $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$.

Putting $z = 0$ in (2) then we see that every H-ideal is an ideal.

Definition 2.4. A fuzzy set μ in a BCI-algebra X is called a fuzzy H-ideal of X if

1. $\mu(0) \ge \mu(x)$,
2. $\mu(x * z) \ge \min\{\mu(x * (y * z)), \mu(y)\}$.

Definition 2.5. An IFS $A = \langle X, \mu_A, \upsilon_A \rangle$ in a BCI-algebra X is called an intuitionistic fuzzy ideal of X if it satisfies:

$(F1)$ $\mu_A(0) \ge \mu_A(x)$ & $\upsilon_A(0) \le \upsilon_A(x)$,

$(F2)$ $\mu_A(x) \ge \min\{\mu_A(x * y), \mu_A(y)\}$,

$(F3)$ $\upsilon_A(x) \le \max\{\upsilon_A(x * y), \upsilon_A(y)\}$, *for all* $x, y \in X$.

Definition 2.6. An intuitionistic fuzzy set $A = \langle \mu_A, \upsilon_A \rangle$ of a BCI-algebra X is called an intuitionistic fuzzy H-ideal if it satisfies $(F1)$ and

$(F4)$ $\mu_A(x * z) \ge \min\{\mu_A(x * (y * z)), \mu_A(y)\}$,

$(F5)\ \upsilon_A(x*z) \leq \max\{\upsilon_A(x*(y*z)), \upsilon_A(y)\}$, for all $x, y, z \in X$.

An interval-valued intuitionistic fuzzy set (briefly, i-v IFS) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)], [\upsilon_A^L(x), \upsilon_A^U(x)])\}, \forall x \in X$$

$(\textit{briefly, denoted by } A = [(\mu_A^L, \mu_A^U), (\upsilon_A^L, \upsilon_A^U)])$, Where μ_A^L, μ_A^U are two membership functions and $\upsilon_A^L, \upsilon_A^U$ are two non-membership functions in X such that $\mu_A^L \leq \mu_A^U \ \& \upsilon_A^L \geq \upsilon_A^U, \forall x \in X$. Let $\overline{\mu}_A(x) = [\mu_A^L, \mu_A^U] \ \& \ \overline{\upsilon}_A(x) = [\upsilon_A^L, \upsilon_A^U], \forall x \in X$ and let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$.

If $\mu_A^L(x) = \mu_A^U(x) = c, 0 \leq c \leq 1$ and if $\upsilon_A^L(x) = \upsilon_A^U(x) = k, 0 \leq k \leq 1$, then we have $\overline{\mu}_A(x) = [c, c] \ \& \ \overline{\upsilon}_A(x) = [k, k]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\overline{\mu}_A(x) \ \& \ \overline{\upsilon}_A(x) \in [0,1], \forall x \in X$, and therefore the i-v IFS A is given by $A = \{(x, \overline{\mu}_A(x), \overline{\upsilon}_A(x))\}, \forall x \in X$, where $\overline{\mu}_A(x) : X \to D[0,1]$, $\overline{\upsilon}_A(x) : X \to D[0,1]$.

Now let us define what is known as refined minimum, refined maximum(briefly rmin, rmax) of two elements in $D[0,1]$. we also define the symbols $"\leq", "\geq"$ *and* $"="$ in the case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$. Then

$$r\min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$$
$$r\max(D_1, D_2) = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$$

$$D_1 \geq D_2 \Leftrightarrow a_1 \geq a_2, b_1 \geq b_2;\ D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2 \text{ and } D_1 = D_2.$$

3. Interval – valued Intuitionistic Fuzzy H-ideals of BCI-algebras

Definition 3.1. An interval-valued intuitionistic fuzzy set A in BCI-algebra X is called an interval-valued intuitionistic fuzzy H-ideal of X if it satisfies

$(FI_1)\ \overline{\mu}_A(0) \geq \overline{\mu}_A(x),\ \overline{\upsilon}_A(0) \leq \overline{\upsilon}_A(x),$

$(FI_2)\ \overline{\mu}_A(x*z) \geq r\min\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\},$

$(FI_3)\ \overline{\upsilon}_A(x*z) \leq r\max\{\overline{\upsilon}_A(x*(y*z)), \overline{\upsilon}_A(y)\}.$

Theorem 3.2. Let A be an i-v intuitionistic fuzzy H-ideal of X. If there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} \overline{\mu}_A(x_n) = [1,1], \lim_{n\to\infty} \overline{\upsilon}_A(x_n) = [0,0]$ then $\overline{\mu}_A(0) = [1,1]$ and $\overline{\upsilon}_A(0) = [0,0]$.

Proof. Since $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$ and $\overline{\upsilon}_A(0) \leq \overline{\upsilon}_A(x)$ for all $x \in X$, we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$ and $\overline{\upsilon}_A(0) \leq \overline{\upsilon}_A(x_n)$, for every positive integer n. Note that

$$[1,1] \geq \bar{\mu}_A(0) \geq \lim_{n\to\infty} \bar{\mu}_A(x_n) = [1,1]; \quad [0,0] \leq \bar{\upsilon}_A(0) \leq \lim_{n\to\infty} \bar{\upsilon}_A(x_n) = [0,0].$$

Hence $\bar{\mu}_A(0) = [1,1]$ and $\bar{\upsilon}_A(0) = [0,0]$.

Lemma 3.3. An i-v intuitionistic fuzzy set $A = \left[\left\langle \mu_A^L, \mu_A^U \right\rangle, \left\langle \upsilon_A^L, \upsilon_A^U \right\rangle\right]$ in X is an i-v intuitionistic fuzzy H-ideal of X if and only if $\left\langle \mu_A^L, \mu_A^U \right\rangle$ and $\left\langle \upsilon_A^L, \upsilon_A^U \right\rangle$ are intuitionistic fuzzy ideals of X .

Proof. Since $\mu_A^L(0) \geq \mu_A^L(x)$; $\mu_A^U(0) \geq \mu_A^U(x)$; $\upsilon_A^L(0) \leq \upsilon_A^L(x)$ and $\upsilon_A^U(0) \leq \upsilon_A^U(x)$, therefore $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$, $\bar{\upsilon}_A(0) \leq \bar{\upsilon}_A(x)$. Suppose that $\left\langle \mu_A^L, \mu_A^U \right\rangle$ and $\left\langle \upsilon_A^L, \upsilon_A^U \right\rangle$ are intuitionistic fuzzy ideal of X. Let $x, y \in X$, then

$$\begin{aligned}
\bar{\mu}_A(x) &= [\mu_A^L(x), \mu_A^U(x)] \\
&\geq \left[\min\left\{\mu_A^L(x*y), \mu_A^L(y)\right\}, \min\left\{\mu_A^U(x*y), \mu_A^U(y)\right\}\right] \\
&= r\min\left\{\left[\mu_A^L(x*y), \mu_A^U(x*y)\right], \left[\mu_A^L(y), \mu_A^U(y)\right]\right\} \\
&= r\min\left\{\bar{\mu}_A(x*y), \bar{\mu}_A(y)\right\} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\bar{\upsilon}_A(x) &= [\upsilon_A^L(x), \upsilon_A^U(x)] \\
&\leq \left[\max\left\{\upsilon_A^L(x*y), \upsilon_A^L(y)\right\}, \max\left\{\upsilon_A^U(x*y), \upsilon_A^U(y)\right\}\right] \\
&= r\max\left\{\left[\upsilon_A^L(x*y), \upsilon_A^U(x*y)\right], \left[\upsilon_A^L(y), \upsilon_A^U(y)\right]\right\} \\
&= r\max\left\{\bar{\upsilon}_A(x*y), \bar{\upsilon}_A(y)\right\}.
\end{aligned}$$

Hence A is an i-v intuitionistic fuzzy ideal of X.

Conversely, assume that A is an i-v intuitionistic fuzzy ideal of X. For any $x, y \in X$, we have

$$\begin{aligned}
\left[\mu_A^L(x), \mu_A^U(x)\right] &= \bar{\mu}_A(x) \\
&\geq r\min\left\{\left[\bar{\mu}_A(x*y), \bar{\mu}_A(y)\right]\right\} \\
&= r\min\left\{\left[\mu_A^L(x*y), \mu_A^U(x*y)\right], \left[\mu_A^L(y), \mu_A^U(y)\right]\right\} \\
&= \left[\min\left\{\mu_A^L(x*y), \mu_A^L(y)\right\}, \min\left\{\mu_A^U(x*y), \mu_A^U(y)\right\}\right]
\end{aligned}$$

and

$$\begin{aligned}
\left[\upsilon_A^L(x), \upsilon_A^U(x)\right] &= \bar{\upsilon}_A(x) \\
&\leq r\max\left\{\left[\bar{\upsilon}_A(x*y), \bar{\upsilon}_A(y)\right]\right\} \\
&= r\max\left\{\left[\upsilon_A^L(x*y), \upsilon_A^U(x*y)\right], \left[\upsilon_A^L(y), \upsilon_A^U(y)\right]\right\} \\
&= \left[\max\left\{\upsilon_A^L(x*y), \upsilon_A^L(y)\right\}, \max\left\{\upsilon_A^U(x*y), \upsilon_A^U(y)\right\}\right]
\end{aligned}$$

It follows that

$$\mu_A^L(x) \geq \min\left\{\mu_A^L(x*y), \mu_A^L(y)\right\}$$
$$\upsilon_A^L(x) \leq \max\left\{\upsilon_A^L(x*y), \upsilon_A^L(y)\right\}$$

and

$$\mu_A^U(x) \geq \min\left\{\mu_A^U(x*y), \mu_A^U(y)\right\}$$
$$\upsilon_A^U(x) \leq \max\left\{\upsilon_A^U(x*y), \upsilon_A^U(y)\right\}$$

Hence $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \upsilon_A^L, \upsilon_A^U \rangle$ are intuitionistic fuzzy ideals of X.

Theorem 3.4. Every i-v intuitionistic fuzzy H-ideal of a BCI-algebra X is an i-v intuitionistic fuzzy ideal.
Proof. Let $A = [\langle \mu_A^L, \mu_A^U \rangle, \langle \upsilon_A^L, \upsilon_A^U \rangle]$ be an i-v intuitionistic fuzzy H-ideal of X, where $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \upsilon_A^L, \upsilon_A^U \rangle$ are intuitionistic fuzzy H-ideals of X. Thus $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \upsilon_A^L, \upsilon_A^U \rangle$ are intuitionistic fuzzy H-ideals of X. Hence by lemma 3.3, A is an i-v intuitionistic fuzzy ideal of X.

Theorem 3.5. Let A be an intuitionistic fuzzy ideal of a BCI-algebra X. If $\overline{\mu}_A(x * y) \geq \overline{\mu}_A(x)$ and $\overline{\upsilon}_A(x * y) \leq \overline{\upsilon}_A(x)$ for all $x, y \in X$, then A is an i-v intuitionistic fuzzy H-ideal of X.
Proof. Since A is an i-v intuitutionistic fuzzy ideal of X, by hypothesis we have
$r\min\{\overline{\mu}_A(x*(y*z)), \overline{\mu}_A(y)\} \leq r\min\{\overline{\mu}_A((x*z)*(y*z)), \overline{\mu}_A(y*z)\} \leq \overline{\mu}_A(x*z)$
and
$r\max\{\overline{\upsilon}_A(x*(y*z)), \overline{\upsilon}_A(y)\} \geq r\max\{\overline{\upsilon}_A((x*z)*(y*z)), \overline{\upsilon}_A(y*z)\} \geq \overline{\upsilon}_A(x*z)$.
For all $x, y, z \in X$. Hence A is an i-v intuitionistic fuzzy H-ideal of X.

Definition 3.6. An i-v intuitionistic fuzzy set A in X is called an interval–valued intuitionistic fuzzy BCI-subalgebra (briefly, i-v IF BCI-subalgebra) of X if $\overline{\mu}_A(x * y) \geq r\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$ and $\overline{\upsilon}_A(x * y) \leq r\max\{\overline{\upsilon}_A(x), \overline{\upsilon}_A(y)\}$, for all $x, y \in X$.

Theorem 3.7. Every i-v intuitionistic fuzzy H-ideal of BCI-algebra X is an i-v intuitionistic fuzzy subalgebra of X.

Proof. Let $A = [\langle \mu_A^L, \mu_A^U \rangle, \langle \upsilon_A^L, \upsilon_A^U \rangle]$ be an i-v intuitionistic fuzzy H-ideal of X, where $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \upsilon_A^L, \upsilon_A^U \rangle$ are intuitionistic fuzzy H-ideals of BCI-algebra X. Thus $\langle \mu_A^L, \mu_A^U \rangle$ and $\langle \upsilon_A^L, \upsilon_A^U \rangle$ are intuitionistic fuzzy subalgebra of X. Hence, A is an i-v intuitionistic fuzzy subalgebra of X.

4. Cartesian product of i– v Intuitionistic Fuzzy H-ideals

Definition 4.1. An intuitionistic fuzzy relation A on any set A is a intuitionistic fuzzy subset A with a membership function $\Omega_A : X \times X \to [0,1]$ and non membership function $\Psi_A : X \times X \to [0,1]$.

Lemma 4.2. Let $\overline{\mu}_A$ and $\overline{\mu}_B$ be two membership functions and $\overline{\upsilon}_A$ and $\overline{\upsilon}_B$ be two non membership functions of each $x \in X$ to the i-v subsets A and B, respectively. Then $\mu_A \times \mu_B$ is membership function and $\upsilon_A \times \upsilon_B$ is non membership function of each element $(x, y) \in X \times X$ to the set $A \times B$ and defined by $(\overline{\mu}_A \times \overline{\mu}_B)(x, y) = r\min\{\overline{\mu}_A(x), \overline{\mu}_B(y)\}$ and $(\overline{\upsilon}_A \times \overline{\upsilon}_B)(x, y) = r\max\{\overline{\upsilon}_A(x), \overline{\upsilon}_B(y)\}$.

Definition 4.3. Let $A = \left[\langle \mu_A^L, \mu_A^U \rangle, \langle \upsilon_A^L, \upsilon_A^U \rangle\right]$ and $B = \left[\langle \mu_B^L, \mu_B^U \rangle, \langle \upsilon_B^L, \upsilon_B^U \rangle\right]$ be two i-v intuitionistic fuzzy subsets in a set X. the Cartesian product of $A \times B$ is defined by

$$A \times B = \left\{\left((x, y), \overline{\mu}_A \times \overline{\mu}_B, \overline{\upsilon}_A \times \overline{\upsilon}_B\right); \forall x, y \in X \times X\right\}$$

where $A \times B : X \times X \to D[0,1]$.

Theorem 4.4. Let $A = \left[\langle \mu_A^L, \mu_A^U \rangle, \langle \upsilon_A^L, \upsilon_A^U \rangle\right]$ and $B = \left[\langle \mu_B^L, \mu_B^U \rangle, \langle \upsilon_B^L, \upsilon_B^U \rangle\right]$ be two i-v intuitionistic fuzzy subsets in a set X, then $A \times B$ is an i-v intuitionistic fuzzy H-ideal of $X \times X$.

Proof. Let $(x, y) \in X \times X$, then by definition

$$\begin{aligned}
\left(\overline{\mu}_A \times \overline{\mu}_B\right)(0,0) &= r\min\{\overline{\mu}_A(0), \overline{\mu}_B(0)\} \\
&= r\min\{[\mu_A^L(0), \mu_A^U(0)], [\mu_B^L(0), \mu_B^U(0)]\} \\
&= \left[\min\{\mu_A^L(0), \mu_B^L(0)\}, \min\{\mu_A^U(0), \mu_B^U(0)\}\right] \\
&\geq \left[\min\{\mu_A^L(x), \mu_B^L(y)\}, \min\{\mu_A^U(x), \mu_B^U(y)\}\right] \\
&= r\min\{[\mu_A^L(x), \mu_A^U(x)], [\mu_B^L(y), \mu_B^U(y)]\} \\
&= r\min\{\overline{\mu}_A(x), \overline{\mu}_B(y)\} \\
&= \left(\overline{\mu}_A \times \overline{\mu}_B\right)(x, y)
\end{aligned}$$

and

$$\begin{aligned}
\left(\overline{\upsilon}_A \times \overline{\upsilon}_B\right)(0,0) &= r\max\{\overline{\upsilon}_A(0), \overline{\upsilon}_B(0)\} \\
&= r\max\{[\upsilon_A^L(0), \upsilon_A^U(0)], [\upsilon_B^L(0), \upsilon_B^U(0)]\} \\
&= \left[\max\{\upsilon_A^L(0), \upsilon_B^L(0)\}, \max\{\upsilon_A^U(0), \upsilon_B^U(0)\}\right] \\
&\leq \left[\max\{\upsilon_A^L(x), \upsilon_B^L(y)\}, \max\{\upsilon_A^U(x), \upsilon_B^U(y)\}\right] \\
&= r\max\{[\upsilon_A^L(x), \upsilon_A^U(x)], [\upsilon_B^L(y), \upsilon_B^U(y)]\} \\
&= r\max\{\overline{\upsilon}_A(x), \overline{\upsilon}_B(y)\} \\
&= \left(\overline{\upsilon}_A \times \overline{\upsilon}_B\right)(x, y)
\end{aligned}$$

Therefore (FI_2) holds.

Now, for all $x, y, z \in X$, we have

$$(\bar{\mu}_A \times \bar{\mu}_B)((x,x')*(z,z')$$
$$= (\bar{\mu}_A \times \bar{\mu}_B)(x*z, x'*z')$$
$$= r\min\{\mu_A(x*z), \mu_B(x'*z')\}$$
$$\geq r\min\left\{r\min\left[\bar{\mu}_A(x*(y*z)), \bar{\mu}_A(y)\right], r\min\left[\bar{\mu}_B(x'*(y'*z')), \bar{\mu}_B(y')\right]\right\}$$
$$= r\min\left\{\begin{array}{l}\left[\min\{\mu_A^L(x*(y*z)), \mu_A^L(y)\}, \min\{\mu_A^U(x*(y*z)), \mu_A^U(y)\}\right], \\ \left[\min\{\mu_B^L(x'*(y'*z')), \mu_B^L(y')\}, \min\{\mu_B^U(x'*(y'*z')), \mu_B^U(y')\}\right]\end{array}\right\}$$

$$= \begin{bmatrix}\min\{\min\{\mu_A^L(x*(y*z)), \mu_B^L(x'*(y'*z'))\}, \min\{\mu_A^L(y), \mu_B^L(y')\}\}, \\ \min\{\min\{\mu_A^U(x*(y*z)), \mu_B^U(x'*(y'*z'))\}, \min\{\mu_A^U(y), \mu_B^U(y')\}\}\end{bmatrix}$$

$$= r\min\left\{\left(\bar{\mu}_A \times \bar{\mu}_B\right)((x*(y*z)), (x'*(y'*z'))), \left(\bar{\mu}_A \times \bar{\mu}_B\right)(y,y')\right\}.$$

Also,

$$(\bar{\upsilon}_A \times \bar{\upsilon}_B)((x,x')*(z,z')$$
$$= (\bar{\upsilon}_A \times \bar{\upsilon}_B)(x*z, x'*z')$$
$$= r\max\{\upsilon_A(x*z), \upsilon_B(x'*z')\}$$
$$\leq r\max\left\{r\max\left[\bar{\upsilon}_A(x*(y*z)), \bar{\upsilon}_A(y)\right], r\max\left[\bar{\upsilon}_B(x'*(y'*z')), \bar{\upsilon}_B(y')\right]\right\}$$
$$= r\max\left\{\begin{array}{l}\left[\max\{\upsilon_A^L(x*(y*z)), \upsilon_A^L(y)\}, \max\{\upsilon_A^U(x*(y*z)), \upsilon_A^U(y)\}\right], \\ \left[\max\{\upsilon_B^L(x'*(y'*z')), \upsilon_B^L(y')\}, \max\{\upsilon_B^U(x'*(y'*z')), \upsilon_B^U(y')\}\right]\end{array}\right\}$$

$$= \begin{bmatrix}\max\{\max\{\upsilon_A^L(x*(y*z)), \upsilon_B^L(x'*(y'*z'))\}, \max\{\upsilon_A^L(y), \upsilon_B^L(y')\}\}, \\ \max\{\max\{\upsilon_A^U(x*(y*z)), \upsilon_B^U(x'*(y'*z'))\}, \max\{\upsilon_A^U(y), \upsilon_B^U(y')\}\}\end{bmatrix}$$

$$= r\max\left\{\left(\bar{\upsilon}_A \times \bar{\upsilon}_B\right)((x*(y*z)), (x'*(y'*z'))), \left(\bar{\upsilon}_A \times \bar{\upsilon}_B\right)(y,y')\right\}.$$

Hence $A \times B$ is an i-v intuitionistic fuzzy H-ideal of $X \times X$.

Definition 4.5. Let $\bar{\mu}_B, \bar{\upsilon}_B$ respectively, be an i-v membership and non membership function of each element $x \in X$ to the set B. Then strongest i-v intuitionistic fuzzy set relation on X, that is a membership function relation $\bar{\mu}_A$ on $\bar{\mu}_B$ and non membership function relation $\bar{\upsilon}_A$ on $\bar{\upsilon}_B$ and $\mu_{A_B}, \upsilon_{A_B}$ whose i-v membership and non membership function, of each element $(x,y) \in X \times X$ and defined by

$$\bar{\mu}_{A_B}(x,y) = r\min\left\{\bar{\mu}_B(x), \bar{\mu}_B(y)\right\} \;\&\; \bar{\upsilon}_{A_B}(x,y) = r\max\left\{\bar{\upsilon}_B(x), \bar{\upsilon}_B(y)\right\}$$

Definition 4.6. Let $B=\left[\left\langle\mu_B^L,\mu_B^U\right\rangle,\left\langle\upsilon_B^L,\upsilon_B^U\right\rangle\right]$ be an i-v subset in a set X. Then the strongest i-v intuitionistic fuzzy relation on X that is a i-v A on B is A_B and defined by,

$$A_B=\left[\left\langle\mu_{A_B}^L,\mu_{A_B}^U\right\rangle,\left\langle\upsilon_{A_B}^L,\upsilon_{A_B}^U\right\rangle\right].$$

Theorem 4.7. Let $B=\left[\left\langle\mu_B^L,\mu_B^U\right\rangle,\left\langle\upsilon_B^L,\upsilon_B^U\right\rangle\right]$ be an i-v subset in a set X and $A_B=\left[\left\langle\mu_{A_B}^L,\mu_{A_B}^U\right\rangle,\left\langle\upsilon_{A_B}^L,\upsilon_{A_B}^U\right\rangle\right]$ be the strongest i-v intuitionistic fuzzy relation on X. Then B is an i-v intuitionistic H-ideal of X if and only if A_B is i-v intuitionistic fuzzy H-ideal of $X\times X$.

Proof. Let B be an i-v intuitionistic fuzzy H-ideal of X. Then

$\overline{\mu}_{A_B}(0,0)=r\min\left\{\overline{\mu}_B(0),\overline{\mu}_B(0)\right\}\geq r\min\left\{\overline{\mu}_B(x),\overline{\mu}_B(y)\right\}=\overline{\mu}_{A_B}(x,y)$ and

$\overline{\upsilon}_{A_B}(0,0)=r\max\left\{\overline{\upsilon}_B(0),\overline{\upsilon}_B(0)\right\}\leq r\max\left\{\overline{\upsilon}_B(x),\overline{\upsilon}_B(y)\right\}=\overline{\upsilon}_{A_B}(x,y)$ for all $(x,y)\in X\times X$.

On the other hand

$$\overline{\mu}_{A_B}((x_1,x_2)*(z_1,z_2))=\overline{\mu}_{A_B}(x_1*z_1,x_2*z_2)$$
$$=r\min\left\{\overline{\mu}_B(x_1*z_1),\overline{\mu}_B(x_2*z_2)\right\}$$
$$\geq r\min\left\{r\min\left\{\overline{\mu}_B(x_1*(y_1*z_1)),\overline{\mu}_B(y_1)\right\},r\min\left\{\overline{\mu}_B(x_2*(y_2*z_2)),\overline{\mu}_B(y_2)\right\}\right\}$$
$$=r\min\left\{r\min\left\{\overline{\mu}_B(x_1*(y_1*z_1)),\overline{\mu}_B(x_2*(y_2*z_2))\right\},r\min\left\{\overline{\mu}_B(y_1),\overline{\mu}_B(y_2)\right\}\right\}$$
$$=r\min\left\{\overline{\mu}_{A_B}(x_1*(y_1*z_1),x_2*(y_2*z_2)),\overline{\mu}_{A_B}(y_1,y_2)\right\}$$
$$=r\min\left\{\overline{\mu}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),\overline{\mu}_{A_B}(y_1,y_2)\right\}$$

Also,

$$\overline{\upsilon}_{A_B}((x_1,x_2)*(z_1,z_2))=\overline{\upsilon}_{A_B}(x_1*z_1,x_2*z_2)$$
$$=r\max\left\{\overline{\upsilon}_B(x_1*z_1),\overline{\upsilon}_B(x_2*z_2)\right\}$$
$$\leq r\max\left\{r\max\left\{\overline{\upsilon}_B(x_1*(y_1*z_1)),\overline{\upsilon}_B(y_1)\right\},r\max\left\{\overline{\upsilon}_B(x_2*(y_2*z_2)),\overline{\upsilon}_B(y_2)\right\}\right\}$$
$$=r\max\left\{r\max\left\{\overline{\upsilon}_B(x_1*(y_1*z_1)),\overline{\upsilon}_B(x_2*(y_2*z_2))\right\},r\max\left\{\overline{\upsilon}_B(y_1),\overline{\upsilon}_B(y_2)\right\}\right\}$$
$$=r\max\left\{\overline{\upsilon}_{A_B}(x_1*(y_1*z_1),x_2*(y_2*z_2)),\overline{\upsilon}_{A_B}(y_1,y_2)\right\}$$
$$=r\max\left\{\overline{\upsilon}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),\overline{\upsilon}_{A_B}(y_1,y_2)\right\}$$

For all $(x_1,x_2),(y_1,y_2),(z_1,z_2)$ in $X\times X$. Hence A_B is an i-v intuitionistic fuzzy H-ideal of $X\times X$.

Conversely, let A_B be an i-v intuitionistic fuzzy H-ideal of $X\times X$. Then for all $(x,x)\in X\times X$, we have

$$r\min\left\{\overline{\mu}_B(0),\overline{\mu}_B(0)\right\}=\overline{\mu}_{A_B}(0,0)\geq\overline{\mu}_{A_B}(x,x)=r\min\left\{\overline{\mu}_B(x),\overline{\mu}_B(x)\right\}$$

(or) $\overline{\mu}_B(0)\geq\overline{\mu}_B(x)$ and

$$r\max\left\{\overline{\upsilon}_B(0),\overline{\upsilon}_B(0)\right\}=\overline{\upsilon}_{A_B}(0,0)\leq\overline{\upsilon}_{A_B}(x,x)=r\max\left\{\overline{\upsilon}_B(x),\overline{\upsilon}_B(x)\right\}$$

(or) $\overline{\upsilon}_B(0)\leq\overline{\upsilon}_B(x)$ for all $x\in X$. Now, let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$, then

$$\begin{aligned}
&\text{rmin}\left\{\overline{\mu}_B(x_1*z_1),\overline{\mu}_B(x_2*z_2)\right\}\\
&=\overline{\mu}_{A_B}(x_1*z_1,x_2*z_2)\\
&=\overline{\mu}_{A_B}((x_1,x_2)*(z_1,z_2))\\
&\geq r\min\left\{\overline{\mu}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),\overline{\mu}_{A_B}(y_1,y_2)\right\}\\
&= r\min\left\{\overline{\mu}_{A_B}(x_1*(y_1*z_1),x_2*(y_2*z_2)),\overline{\mu}_{A_B}(y_1,y_2)\right\}
\end{aligned}$$

$$=r\min\left\{r\min\left\{\overline{\mu}_B(x_1*(y_1*z_1)),\overline{\mu}_B(y_1)\right\},r\min\left\{\overline{\mu}_B(x_2*(y_2*z_2)),\overline{\mu}_B(y_2)\right\}\right\}$$

Also,

$$\begin{aligned}
&\text{rmax}\left\{\overline{\upsilon}_B(x_1*z_1),\overline{\upsilon}_B(x_2*z_2)\right\}\\
&=\overline{\upsilon}_{A_B}(x_1*z_1,x_2*z_2)\\
&=\overline{\upsilon}_{A_B}((x_1,x_2)*(z_1,z_2))\\
&\leq r\max\left\{\overline{\upsilon}_{A_B}((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),\overline{\upsilon}_{A_B}(y_1,y_2)\right\}\\
&= r\max\left\{\overline{\upsilon}_{A_B}(x_1*(y_1*z_1),x_2*(y_2*z_2)),\overline{\upsilon}_{A_B}(y_1,y_2)\right\}
\end{aligned}$$

$$=r\max\left\{r\max\left\{\overline{\upsilon}_B(x_1*(y_1*z_1)),\overline{\upsilon}_B(y_1)\right\},r\max\left\{\overline{\upsilon}_B(x_2*(y_2*z_2)),\overline{\upsilon}_B(y_2)\right\}\right\}$$

If $x_2=y_2=z_2=0$, then

$r\min\left\{\overline{\mu}_B(x_1*z_1),\overline{\mu}_B(0)\right\}\geq r\min\left\{r\min\left\{\overline{\mu}_B(x_1*(y_1*z_1)),\overline{\mu}_B(y_1)\right\},\overline{\mu}_B(0)\right\}$ and

$r\max\left\{\overline{\upsilon}_B(x_1*z_1),\overline{\upsilon}_B(0)\right\}\leq r\max\left\{r\max\left\{\overline{\upsilon}_B(x_1*(y_1*z_1)),\overline{\upsilon}_B(y_1)\right\},\overline{\upsilon}_B(0)\right\}$ or

$\overline{\mu}_B(x_1*z_1)\geq r\min\left\{\overline{\mu}_B(x_1*(y_1*z_1)),\overline{\mu}_B(y_1)\right\}$ and

$\overline{\upsilon}_B(x_1*z_1)\leq r\max\left\{\overline{\upsilon}_B(x_1*(y_1*z_1)),\overline{\upsilon}_B(y_1)\right\}$.

Therefore B is an i-v intuitionistic fuzzy H-ideal of X.

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