Intuitionistic Fuzzy H-Ideals of BCI-Algebras

with Interval Valued Membership

& Non Membership Functions

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Abstract. The purpose of this paper is to define the notion of an interval valued intuitionistic fuzzy H-ideal (briefly, an i-v IF H-ideal) of a BCI-algebra. Necessary and sufficient conditions for an i-v intuitionistic fuzzy H-ideal are stated. Cartesian product of i-v intuitionistic fuzzy ideals are discussed.

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1. Introduction

The notion of BCK-algebras was proposed by Imai and Iseki in 1966. In the same year, Iseki [6] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [10]. In [9], Zadeh made an
extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. In [9], Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. In [4], Biswas defined interval-valued fuzzy subgroups (i.e., i-v fuzzy subgroups) of Rosenfeld’s nature, and investigated some elementary properties. The idea of “intuitionistic fuzzy set” was first published by Atanassov[1],[2] as a generalization of the notion of fuzzy sets. After that many researchers considered the fuzzification of ideals and subalgebras in BCK(BCI)-algebras. In this paper, using the notion of interval-valued fuzzy set, we introduce the concept of an interval-valued intuitionistic fuzzy BCI-algebra (briefly, i-v IF subalgebra) of a BCI-algebra, and study some of their properties. Using an i-v level set of an i-v intuitionistic fuzzy set, we state a characterization of an intuitionistic fuzzy H-ideal of BCI-algebras. We prove that every intuitionistic fuzzy H-ideal of a BCI-algebra X can be realized as an i-v level H-ideal of an i-v intuitionistic fuzzy H-ideal of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v intuitionistic fuzzy H-ideal become i-v intuitionistic fuzzy H-ideal.

2. Preliminaries

Let us recall that an algebra \((X, *, 0)\) of type \((2, 0)\) is called a BCI-algebra if it satisfies the following conditions:
1. \(((x * y) * (x * z)) * (z * y) = 0\),
2. \((x * (x * y)) * y = 0\),
3. \(x * x = 0\),
4. \(x * y = 0\) and \(y * x = 0\) imply \(x = y\), for all \(x, y, z \in X\).

In a BCI-algebra, we can define a partial ordering "\(\leq\)" by \(x \leq y\) if and only if \(x * y = 0\). In a BCI-algebra \(X\), the set \(M = \{x \in X \mid 0 * x = 0\}\) is a subalgebra and is called the BCK-part of \(X\). A BCI-algebra \(X\) is called proper if \(X - M \neq \phi\). Otherwise it is improper. Moreover, in a BCI-algebra the following conditions hold:
5. \((x * y) * z = (x * z) * y\),
6. \(x * 0 = 0\),
7. \(x \leq y\) imply \(x * z \leq y * z\) and \(z * y \leq z * x\),
8. \(0 * (x * y) = (0 * x) * (0 * y)\),
9. \(0 * (0 * (x * y)) = 0 * (y * x)\),
10. \((x * z) * (y * z) \leq x * y\).

An intuitionistic fuzzy set \(A\) in a non-empty set \(X\) is an object having the form
\[
A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \},
\]
where the functions $\mu_A : X \to [0,1]$ and $\upsilon_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\upsilon_A(x)$) of each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for Example two journals:
1. Fuzzy Sets and 2. Notes on Intuitionistic Fuzzy Sets) and have many interesting applications not only in mathematics (See Chapter 5 in the book [2]).

For the sake of simplicity, we shall use the symbol $A = \{ \mu_A, \upsilon_A \}$ for the intuitionistic fuzzy set $A = \{ \{x, \mu_A(x), \upsilon_A(x)\} | x \in X \}$

**Definition 2.1.** A non empty subset $I$ of $X$ is called an ideal of $X$ if it satisfies:
1. $0 \in I$,
2. $x \ast y \in I$ and $y \in I$ imply $x \in I$.

**Definition 2.2.** A fuzzy subset $\mu$ of a BCI-algebra $X$ is called an fuzzy ideal of $X$ if it satisfies:
1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$, for all $x, y \in X$.

**Definition 2.3.** A non empty subset $I$ of $X$ is called an H-ideal of $X$ if it satisfies:
1. $0 \in I$,
2. $x \ast (y \ast z) \in I$ and $y \in I$ imply $x \ast z \in I$.

Putting $z = 0$ in (2) then we see that every H-ideal is an ideal.

**Definition 2.4.** A fuzzy set $\mu$ in a BCI-algebra $X$ is called a fuzzy H-ideal of $X$ if it satisfies:
1. $\mu(0) \geq \mu(x)$,
2. $\mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(y)\}$.

**Definition 2.5.** An IFS $A = \{ X, \mu_A, \upsilon_A \}$ in a BCI-algebra $X$ is called an intuitionistic fuzzy ideal of $X$ if it satisfies:
(F1) $\mu_A(0) \geq \mu_A(x) \& \upsilon_A(0) \leq \upsilon_A(x)$,
(F2) $\mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(y)\}$,
(F3) $\upsilon_A(x) \leq \max\{\upsilon_A(x \ast y), \upsilon_A(y)\}$, for all $x, y \in X$.

**Definition 2.6.** An intuitionistic fuzzy set $A = \{ \mu_A, \upsilon_A \}$ of a BCI-algebra $X$ is called an intuitionistic fuzzy H-ideal if it satisfies (F1) and
(F4) $\mu_A(x \ast z) \geq \min\{\mu_A(x \ast (y \ast z)), \mu_A(y)\}$,
An interval-valued intuitionistic fuzzy set (briefly, i-v IFS) $A$ defined on $X$ is given by

$$A = \left\{ \left(x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)] \right) \right\}, \forall x \in X$$

(briefly, denoted by $A = [(\mu_A^L, \mu_A^U), (\nu_A^L, \nu_A^U)]$), where $\mu_A^L, \mu_A^U$ are two membership functions and $\nu_A^L, \nu_A^U$ are two non-membership functions in $X$ such that $\mu_A^L \leq \mu_A^U$ and $\nu_A^L \leq \nu_A^U$, $\forall x \in X$. Let $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ and $\bar{\nu}_A(x) = [\nu_A^L(x), \nu_A^U(x)]$, $\forall x \in X$ and let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$.

If $\mu_A^L(x) = \mu_A^U(x) = c, 0 \leq c \leq 1$ and if $\nu_A^L(x) = \nu_A^U(x) = k, 0 \leq k \leq 1$, then we have $\bar{\mu}_A(x) = [c, c]$ and $\bar{\nu}_A(x) = [k, k]$ which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\bar{\mu}_A(x) \in D[0,1], \forall x \in X$ and therefore the i-v IFS $A$ is given by $A = \left\{ (x, \bar{\mu}_A(x), \bar{\nu}_A(x)) \right\}, \forall x \in X$.

where $\bar{\mu}_A(x) : X \to D[0,1], \bar{\nu}_A(x) : X \to D[0,1]$.

Now let us define what is known as refined minimum, refined maximum(briefly rmin, rmax) of two elements in $D[0,1]$. we also define the symbols "\leq", "\geq", "\min", "\max" in the case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0,1]$. Then

$$r \min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$$

$$r \max(D_1, D_2) = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$$

Then $D_1 \geq D_2 \iff a_1 \geq a_2, b_1 \geq b_2$; $D_1 \leq D_2 \iff a_1 \leq a_2, b_1 \leq b_2$ and $D_1 = D_2$.

3. Interval-valued Intuitionistic Fuzzy H-ideals of BCI-algebras

**Definition 3.1.** An interval-valued intuitionistic fuzzy set $A$ in BCI-algebra $X$ is called an interval-valued intuitionistic fuzzy H-ideal of $X$ if it satisfies

$$(FI_1) \bar{\mu}_A(0) \geq \bar{\mu}_A(x), \bar{\nu}_A(0) \leq \bar{\nu}_A(x),$$

$$(FI_2) \bar{\mu}_A(x \ast z) \geq \max\{\bar{\mu}_A(x \ast (y \ast z)), \bar{\mu}_A(y)\},$$

$$(FI_3) \bar{\nu}_A(x \ast z) \leq \min\{\bar{\nu}_A(x \ast (y \ast z)), \bar{\nu}_A(y)\}.$$
Intuitionistic fuzzy H-ideals

\[ [1,1] \geq \bar{\mu}_A(0) \geq \lim_{n \to \infty} \bar{\mu}_A(x_n) = [1,1]; \hspace{0.5cm} [0,0] \leq \bar{\nu}_A(0) \leq \lim_{n \to \infty} \bar{\nu}_A(x_n) = [0,0]. \]

Hence \( \bar{\mu}_A(0) = [1,1] \) and \( \bar{\nu}_A(0) = [0,0] \).

**Lemma 3.3.** An i-v intuitionistic fuzzy set \( A = [\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U] \) in \( X \) is an i-v intuitionistic fuzzy H-ideal of \( X \) if and only if \( \{\mu_A^L, \mu_A^U\} \) and \( \{\nu_A^L, \nu_A^U\} \) are intuitionistic fuzzy ideals of \( X \).

**Proof.** Since \( \mu_A^L(0) \geq \mu_A^L(x) ; \mu_A^U(0) \geq \mu_A^U(x) ; \nu_A^L(0) \leq \nu_A^L(x) \) and \( \nu_A^U(0) \leq \nu_A^U(x) \), therefore \( \bar{\mu}_A(0) \geq \bar{\mu}_A(x) \), \( \bar{\nu}_A(0) \leq \bar{\nu}_A(x) \). Suppose that \( \{\mu_A^L, \mu_A^U\} \) and \( \{\nu_A^L, \nu_A^U\} \) are intuitionistic fuzzy ideal of \( X \). Let \( x, y \in X \), then

\[
\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)] \\
\geq \left[ \min \{\mu_A^L(x \ast y), \mu_A^L(y)\}, \min \{\mu_A^U(x \ast y), \mu_A^U(y)\} \right] \\
= r \min \left[ \left[ \mu_A^L(x \ast y), \mu_A^L(x \ast y) \right], \left[ \mu_A^U(x \ast y), \mu_A^U(x \ast y) \right] \right] \\
= r \min \left[ \mu_A^L(x \ast y), \mu_A^L(y) \right] \text{ and} \\
\bar{\nu}_A(x) = [\nu_A^L(x), \nu_A^U(x)] \\
\leq \left[ \max \{\nu_A^L(x \ast y), \nu_A^L(y)\}, \max \{\nu_A^U(x \ast y), \nu_A^U(y)\} \right] \\
= r \max \left[ \left[ \nu_A^L(x \ast y), \nu_A^L(x \ast y) \right], \left[ \nu_A^U(x \ast y), \nu_A^U(x \ast y) \right] \right] \\
= r \max \left[ \nu_A^L(x \ast y), \nu_A^L(y) \right].
\]

Hence \( A \) is an i-v intuitionistic fuzzy ideal of \( X \).

Conversely, assume that \( A \) is an i-v intuitionistic fuzzy ideal of \( X \). For any \( x, y \in X \), we have

\[
[\mu_A^L(x), \mu_A^U(x)] = \bar{\mu}_A(x) \\
\geq r \min \left[ \left[ \mu_A^L(x \ast y), \mu_A^L(y) \right] \right] \\
= r \min \left[ \left[ \mu_A^L(x \ast y), \mu_A^L(x \ast y) \right], \left[ \mu_A^U(x \ast y), \mu_A^U(x \ast y) \right] \right] \\
= \left[ \min \{\mu_A^L(x \ast y), \mu_A^L(y)\}, \min \{\mu_A^U(x \ast y), \mu_A^U(y)\} \right]
\]

and

\[
[\nu_A^L(x), \nu_A^U(x)] = \bar{\nu}_A(x) \\
\leq r \max \left[ \left[ \nu_A^L(x \ast y), \nu_A^L(y) \right] \right] \\
= r \max \left[ \left[ \nu_A^L(x \ast y), \nu_A^L(x \ast y) \right], \left[ \nu_A^U(x \ast y), \nu_A^U(x \ast y) \right] \right] \\
= \left[ \max \{\nu_A^L(x \ast y), \nu_A^L(y)\}, \max \{\nu_A^U(x \ast y), \nu_A^U(y)\} \right]
\]

It follows that

\[
\mu_A^L(x) \geq \min \{\mu_A^L(x \ast y), \mu_A^L(y)\} \\
\nu_A^L(x) \leq \max \{\nu_A^L(x \ast y), \nu_A^L(y)\}
\]

and

\[
\mu_A^U(x) \geq \min \{\mu_A^U(x \ast y), \mu_A^U(y)\} \\
\nu_A^U(x) \leq \max \{\nu_A^U(x \ast y), \nu_A^U(y)\}
\]
Hence \( \langle \mu_A^+, \mu_A^- \rangle \) and \( \langle \nu_A^+, \nu_A^- \rangle \) are intuitionistic fuzzy ideals of \( X \).

**Theorem 3.4.** Every i-v intuitionistic fuzzy H-ideal of a BCI-algebra \( X \) is an i-v intuitionistic fuzzy ideal.

**Proof.** Let \( A = \langle \mu_A^+, \mu_A^- \rangle, \langle \nu_A^+, \nu_A^- \rangle \rangle \) be an i-v intuitionistic fuzzy H-ideal of \( X \), where \( \langle \mu_A^+, \mu_A^- \rangle \) and \( \langle \nu_A^+, \nu_A^- \rangle \) are intuitionistic fuzzy H-ideals of \( X \).

Thus \( \langle \mu_A^+, \mu_A^- \rangle \) and \( \langle \nu_A^+, \nu_A^- \rangle \) are intuitionistic fuzzy H-ideals of \( X \). Hence by lemma 3.3, \( A \) is an i-v intuitionistic fuzzy ideal of \( X \).

**Theorem 3.5.** Let \( A \) be an intuitionistic fuzzy ideal of a BCI-algebra \( X \).

If \( \mu_A(x \cdot y) \geq \mu_A(x) \) and \( \nu_A(x \cdot y) \leq \nu_A(x) \) for all \( x, y \in X \), then \( A \) is an i-v intuitionistic fuzzy H-ideal of \( X \).

**Proof.** Since \( A \) is an i-v intuitionistic fuzzy ideal of \( X \), by hypothesis we have

\[
\begin{align*}
\min \left\{ \mu_A(x \cdot (y \cdot z)), \mu_A(y) \right\} &\leq \min \left\{ \mu_A((x \cdot z) \cdot (y \cdot z)), \mu_A(y \cdot z) \right\} \\
\min \left\{ \nu_A(x \cdot (y \cdot z)), \mu_A(y) \right\} &\leq \min \left\{ \nu_A((x \cdot z) \cdot (y \cdot z)), \nu_A(y \cdot z) \right\}
\end{align*}
\]

and

\[
\begin{align*}
\max \left\{ \nu_A(x \cdot (y \cdot z)), \nu_A(y) \right\} &\geq \max \left\{ \nu_A((x \cdot z) \cdot (y \cdot z)), \nu_A(y \cdot z) \right\} \\
\max \left\{ \nu_A(x \cdot (y \cdot z)), \nu_A(y) \right\} &\geq \max \left\{ \nu_A((x \cdot z) \cdot (y \cdot z)), \nu_A(y \cdot z) \right\}
\end{align*}
\]

For all \( x, y, z \in X \). Hence \( A \) is an i-v intuitionistic fuzzy H-ideal of \( X \).

**Definition 3.6.** An i-v intuitionistic fuzzy set \( A \) in \( X \) is called an interval-valued intuitionistic fuzzy BCI-subalgebra (briefly, i-v IF BCI-subalgebra) of \( X \) if \( \mu_A(x \cdot y) \geq \mu_A(x) \) and \( \nu_A(x \cdot y) \leq \nu_A(x) \) for all \( x, y \in X \).

**Theorem 3.7.** Every i-v intuitionistic fuzzy H-ideal of BCI-algebra \( X \) is an i-v intuitionistic fuzzy subalgebra of \( X \).

**Proof.** Let \( A = \langle \mu_A^+, \mu_A^- \rangle, \langle \nu_A^+, \nu_A^- \rangle \rangle \) be an i-v intuitionistic fuzzy H-ideal of \( X \), where \( \langle \mu_A^+, \mu_A^- \rangle \) and \( \langle \nu_A^+, \nu_A^- \rangle \) are intuitionistic fuzzy H-ideals of BCI-algebra \( X \). Thus \( \langle \mu_A^+, \mu_A^- \rangle \) and \( \langle \nu_A^+, \nu_A^- \rangle \) are intuitionistic fuzzy subalgebra of \( X \). Hence, \( A \) is an i-v intuitionistic fuzzy subalgebra of \( X \).

4. Cartesian product of i-v Intuitionistic Fuzzy H-ideals

**Definition 4.1.** An intuitionistic fuzzy relation \( A \) on any set \( X \) is an intuitionistic fuzzy subset \( A \) with a membership function \( \Omega_A : X \times X \rightarrow [0,1] \) and non membership function \( \Psi_A : X \times X \rightarrow [0,1] \).
Lemma 4.2. Let $\mu_A$ and $\mu_B$ be two membership functions and $\nu_A$ and $\nu_B$ be two non membership functions of each $x \in X$ to the i-v subsets $A$ and $B$, respectively. Then $\mu_A \times \mu_B$ is membership function and $\nu_A \times \nu_B$ is non membership function of each element $(x,y) \in X \times X$ to the set $A \times B$ and defined by $\overline{\mu_A \times \mu_B}(x,y) = r \min \{\mu_A(x) \times \mu_B(y)\}$ and $\overline{\nu_A \times \nu_B}(x,y) = r \max \{\nu_A(x) \times \nu_B(y)\}$.

Definition 4.3. Let $A = [\mu_A^L, \mu_A^U, \nu_A^L, \nu_A^U]$ and $B = [\mu_B^L, \mu_B^U, \nu_B^L, \nu_B^U]$ be two i-v intuitionistic fuzzy subsets in a set $X$. The Cartesian product of $A \times B$ is defined by $A \times B = \{(x,y) \in A \times B, \forall x,y \in X \times X \}$ where $A \times B : X \times X \to D[0,1]$.

Theorem 4.4. Let $A = [\mu_A^L, \mu_A^U, \nu_A^L, \nu_A^U]$ and $B = [\mu_B^L, \mu_B^U, \nu_B^L, \nu_B^U]$ be two i-v intuitionistic fuzzy subsets in a set $X$, then $A \times B$ is an i-v intuitionistic fuzzy $H$-ideal of $X \times X$.

Proof. Let $(x,y) \in X \times X$, then by definition

$$\overline{\mu_A \times \mu_B}(x,y) = r \min \{\mu_A(0) \times \mu_B(0)\} = r \min \{\mu_A^L(0), \mu_A^U(0), \mu_B^L(0), \mu_B^U(0)\} \leq \min \{\mu_A^L(x), \mu_A^U(x), \mu_B^L(y), \mu_B^U(y)\} = r \min \{\mu_A(x), \mu_B(y)\} = (\overline{\mu_A \times \mu_B})(x,y)$$

and

$$\overline{\nu_A \times \nu_B}(x,y) = r \max \{\nu_A(0) \times \nu_B(0)\} = r \max \{\nu_A^L(0), \nu_A^U(0), \nu_B^L(0), \nu_B^U(0)\}\leq \max \{\nu_A^L(x), \nu_A^U(x), \nu_B^L(y), \nu_B^U(y)\} = r \max \{\nu_A(x), \nu_B(y)\} = (\overline{\nu_A \times \nu_B})(x,y)$$

Therefore $(FI_2)$ holds.

Now, for all $x, y, z \in X$, we have
\[(\overline{\mu}_A \times \overline{\mu}_B)((x, x') \ast (z, z'))\]
\begin{align*}
&= (\overline{\mu}_A \times \overline{\mu}_B)(x \ast z, x' \ast z') \\
&= r \min \{\mu_A(x \ast z), \mu_B(x' \ast z')\} \\
&\geq r \min \left\{r \min \left[\mu_A(x \ast (y \ast z)), \mu_B(y)\right], r \min \left[\mu_B(x' \ast (y' \ast z')), \mu_B(y')\right]\right\} \\
&= r \min \left\{\left[\min \left[\mu_A^L(x \ast (y \ast z)), \mu_A^U(y)\right], \min \left[\mu_B^L(x' \ast (y' \ast z')), \mu_B^U(y')\right]\right] \right\} \\
&= \left[\min \left[\min \left[\mu_A^L(x \ast (y \ast z)), \mu_B^L(x' \ast (y' \ast z'))\right], \min \left[\mu_A^U(y), \mu_B^U(y')\right]\right] \right\} \\
&= \left[\min \left[\min \left[\mu_A^L(x \ast (y \ast z)), \mu_B^L(x' \ast (y' \ast z'))\right], \min \left[\mu_A^U(y), \mu_B^U(y')\right]\right] \right\} \\
&= r \min \left\{\left(\mu_A \times \overline{\mu}_B\right)((x, (y \ast z)), (x', (y' \ast z'))), \left(\mu_A \times \overline{\mu}_B\right)(y, y')\right\}.
\end{align*}

Also,
\[(\overline{\nu}_A \times \overline{\nu}_B)((x, x') \ast (z, z'))\]
\begin{align*}
&= (\overline{\nu}_A \times \overline{\nu}_B)(x \ast z, x' \ast z') \\
&= r \max \{\nu_A(x \ast z), \nu_B(x' \ast z')\} \\
&\leq r \max \left\{r \max \left[\nu_A(x \ast (y \ast z)), \nu_A(y)\right], r \max \left[\nu_B(x' \ast (y' \ast z')), \nu_B(y')\right]\right\} \\
&= r \max \left\{\left[\max \left[\nu_A^L(x \ast (y \ast z)), \nu_A^U(y)\right], \max \left[\nu_B^L(x' \ast (y' \ast z')), \nu_B^U(y')\right]\right] \right\} \\
&= \left[\max \left[\max \left[\nu_A^L(x \ast (y \ast z)), \nu_B^L(x' \ast (y' \ast z'))\right], \max \left[\nu_A^U(y), \nu_B^U(y')\right]\right] \right\} \\
&= \left[\max \left[\max \left[\nu_A^L(x \ast (y \ast z)), \nu_B^L(x' \ast (y' \ast z'))\right], \max \left[\nu_A^U(y), \nu_B^U(y')\right]\right] \right\} \\
&= r \max \left\{\left(\overline{\nu}_A \times \overline{\nu}_B\right)((x, (y \ast z)), (x', (y' \ast z'))), \left(\overline{\nu}_A \times \overline{\nu}_B\right)(y, y')\right\}.
\end{align*}

Hence \(A \times B\) is an i-v intuitionistic fuzzy H-ideal of \(X \times X\).

**Definition 4.5.** Let \(\overline{\mu}_B, \overline{\nu}_B\) respectively, be an i-v membership and non membership function of each element \(x \in X\) to the set \(B\). Then strongest i-v intuitionistic fuzzy set relation on \(X\), that is a membership function relation \(\overline{\mu}_A\) on \(\overline{\mu}_B\) and non membership function relation \(\overline{\nu}_A\) on \(\overline{\nu}_B\) and \(\mu_A, \nu_A\) whose i-v membership and non membership function, of each element \((x, y) \in X \times X\) and defined by
\[\overline{\mu}_{\mu_A}(x, y) = r \min \left\{\mu_B(x), \overline{\mu}_B(y)\right\} \quad & \quad \overline{\nu}_{\nu_A}(x, y) = r \max \left\{\overline{\nu}_B(x), \nu_B(y)\right\}\]
**Definition 4.6.** Let $B = \left[\left(\mu^L_B, \mu^U_B\right), \left(\nu^L_B, \nu^U_B\right)\right]$ be an i-v subset in a set $X$. Then the strongest i-v intuitionistic fuzzy relation on $X$ that is a i-v $A$ on $B$ is $A_B$ and defined by,

$$A_B = \left[\left(\mu^L_B, \mu^U_B\right), \left(\nu^L_B, \nu^U_B\right)\right]$$

**Theorem 4.7.** Let $B = \left[\left(\mu^L_B, \mu^U_B\right), \left(\nu^L_B, \nu^U_B\right)\right]$ be an i-v subset in a set $X$ and $A_B = \left[\left(\mu^L_B, \mu^U_B\right), \left(\nu^L_B, \nu^U_B\right)\right]$ be the strongest i-v intuitionistic fuzzy relation on $X$. Then $B$ is an i-v intuitionistic H-ideal of $X$ if and only if $A_B$ is i-v intuitionistic fuzzy H-ideal of $X \times X$.

**Proof.** Let $B$ be an i-v intuitionistic fuzzy H-ideal of $X$. Then

$$\mu_{A_B}(0,0) = r \min\{\mu_B(0)\geq r \min\{\mu_B(x), \mu_B(y)\} = \mu_{A_B}(x, y) \text{ and}$$

$$\nu_{A_B}(0,0) = r \max\{\nu_B(0)\leq r \max\{\nu_B(x), \nu_B(y)\} = \nu_{A_B}(x, y) \text{ for all } (x, y) \in X \times X.$$ 

On the other hand

$$\mu_{A_B}((x_1, x_2) * (z_1, z_2)) = \mu_{A_B}(x_1 * z_1, x_2 * z_2)$$

$$= r \min\{\mu_B(x_1 * z_1), \mu_B(x_2 * z_2)\}$$

$$\geq r \min\{r \min\{\mu_B(x_1 * (y_1 * z_1)), \mu_B(y_1)\}, r \min\{\mu_B(x_2 * (y_2 * z_2)), \mu_B(y_2)\}\}$$

$$= r \min\{\mu_B(x_1 * (y_1 * z_1)), \mu_B(x_2 * (y_2 * z_2))\} r \min\{\mu_B(y_1), \mu_B(y_2)\}$$

$$= r \min\{\mu_{A_B}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2))\, \mu_{A_B}(y_1, y_2)\}$$

$$= r \min\{\mu_{A_B}((x_1, x_2) * ((y_1, y_2) * (z_1, z_2)))\, \mu_{A_B}(y_1, y_2)\}$$

Also,

$$\nu_{A_B}((x_1, x_2) * (z_1, z_2)) = \nu_{A_B}(x_1 * z_1, x_2 * z_2)$$

$$= r \max\{\nu_B(x_1 * z_1), \nu_B(x_2 * z_2)\}$$

$$\leq r \max\{r \max\{\nu_B(x_1 * (y_1 * z_1)), \nu_B(y_1)\}, r \max\{\nu_B(x_2 * (y_2 * z_2)), \nu_B(y_2)\}\}$$

$$= r \max\{r \max\{\nu_B(x_1 * (y_1 * z_1)), \nu_B(x_2 * (y_2 * z_2))\} r \max\{\nu_B(y_1), \nu_B(y_2)\}\}$$

$$= r \max\{\nu_B(x_1 * (y_1 * z_1)), \nu_B(x_2 * (y_2 * z_2))\} r \max\{\nu_B(y_1), \nu_B(y_2)\}$$

$$= r \max\{\nu_{A_B}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2))\, \nu_{A_B}(y_1, y_2)\}$$

$$= r \max\{\nu_{A_B}((x_1, x_2) * ((y_1, y_2) * (z_1, z_2)))\, \nu_{A_B}(y_1, y_2)\}$$

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ in $X \times X$. Hence $A_B$ is an i-v intuitionistic fuzzy H-ideal of $X \times X$.

Conversely, let $A_B$ be an i-v intuitionistic fuzzy H-ideal of $X \times X$. Then for all $(x, x) \in X \times X$, we have
\[ r \min \left\{ \mu_B(0), \mu_B(0) \right\} = \mu_{\mathcal{A}_B}(0,0) \geq \mu_{\mathcal{A}_B}(x,x) = r \min \left\{ \mu_B(x), \mu_B(x) \right\} \]
(or) \( \mu_B(0) \geq \mu_{\mathcal{A}_B}(x) \) and
\[ r \max \left\{ \nu_B(0), \nu_B(0) \right\} = \nu_{\mathcal{A}_B}(0,0) \leq \nu_{\mathcal{A}_B}(x,x) = r \max \left\{ \nu_B(x), \nu_B(x) \right\} \]
(or) \( \nu_B(0) \leq \nu_{\mathcal{A}_B}(x) \) for all \( x \in X \). Now, let \( (x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \), then
\[ r \min \left\{ \mu_B(x_1 \ast z_1), \mu_B(x_2 \ast z_2) \right\} \]
\[ = \mu_{\mathcal{A}_B}(x_1 \ast z_1, x_2 \ast z_2) \]
\[ = \mu_{\mathcal{A}_B}(x_1, x_2) \ast (z_1, z_2) \]
\[ \geq r \min \left\{ \mu_{\mathcal{A}_B}((x_1, x_2) \ast ((y_1, y_2) \ast (z_1, z_2))), \mu_{\mathcal{A}_B}(y_1, y_2) \right\} \]
\[ = r \min \left\{ \mu_{\mathcal{A}_B}(x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2)), \mu_{\mathcal{A}_B}(y_1, y_2) \right\} \]
\[ = r \min \left\{ r \min \left\{ \mu_B(x_1 \ast (y_1 \ast z_1)), \mu_B(y_1) \right\}, r \min \left\{ \mu_B(x_2 \ast (y_2 \ast z_2)), \mu_B(y_2) \right\} \right\} \]
Also,
\[ r \max \left\{ \nu_B(x_1 \ast z_1), \nu_B(x_2 \ast z_2) \right\} \]
\[ = \nu_{\mathcal{A}_B}(x_1 \ast z_1, x_2 \ast z_2) \]
\[ = \nu_{\mathcal{A}_B}(x_1, x_2) \ast (z_1, z_2) \]
\[ \leq r \max \left\{ \nu_{\mathcal{A}_B}((x_1, x_2) \ast ((y_1, y_2) \ast (z_1, z_2))), \nu_{\mathcal{A}_B}(y_1, y_2) \right\} \]
\[ = r \max \left\{ \nu_{\mathcal{A}_B}(x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2)), \nu_{\mathcal{A}_B}(y_1, y_2) \right\} \]
\[ = r \max \left\{ r \max \left\{ \nu_B(x_1 \ast (y_1 \ast z_1)), \nu_B(y_1) \right\}, r \max \left\{ \nu_B(x_2 \ast (y_2 \ast z_2)), \nu_B(y_2) \right\} \right\} \]
If \( x_2 = y_2 = z_2 = 0 \), then
\[ r \min \left\{ \mu_B(x_1 \ast z_1), \mu_B(0) \right\} \geq r \min \left\{ r \min \left\{ \mu_B(x_1 \ast (y_1 \ast z_1)), \mu_B(y_1) \right\}, \mu_B(0) \right\} \]
and
\[ r \max \left\{ \nu_B(x_1 \ast z_1), \nu_B(0) \right\} \leq r \max \left\{ r \max \left\{ \nu_B(x_1 \ast (y_1 \ast z_1)), \nu_B(y_1) \right\}, \nu_B(0) \right\} \]
or
\[ \mu_B(x_1 \ast z_1) \geq r \min \left\{ \nu_B(x_1 \ast (y_1 \ast z_1)), \nu_B(y_1) \right\} \]
and
\[ \nu_B(x_1 \ast z_1) \leq r \max \left\{ \nu_B(x_1 \ast (y_1 \ast z_1)), \nu_B(y_1) \right\} . \]
Therefore \( B \) is an \( i-v \) intuitionistic fuzzy \( H \)-ideal of \( X \).

References


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