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t-Balancing Numbers

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Abstract

A positive integer *n* is called a *co-balancing number* [8] if

$$1 + 2 + \cdots \ldots \ldots + n = (n+1) + (n+2) + \ldots \ldots \ldots + (n+r)$$

for some positive integer r called the *co-balancer* and *balancing number* [1] if

$$1 + 2 + \cdots \ldots \ldots + n - 1 = (n+1) + (n+2) + \ldots \ldots \ldots + (n+r)$$

where *r* is called *balancer*. Since the first paper published on *balancing number* by A. Behera and G. K. Panda[1], several researchers have contributed a lot of interesting results on these two numbers. Panda [6] and Liptai [2] have generalized the results to different sequences, particularly the sequence of the type, $1^k + 2^k + \cdots + n^k$.

Here we have tried to generalize it in a innovative way.

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1. INTRODUCTION

A positive integer *n* is called a *t-balancing number* if

$$1+2+\cdots\ldots\ldots+n = (n+1+t)+(n+2+t)+\ \ldots\ldots\ldots+(n+r+t)$$

for some positive integer *r*, called the *t-balancer*. We denote the n^{th} *t-balancing number* by B_n^t and n^{th} *t-balancer* by R_n^t .

The following are some examples of *t-balancing numbers* for different values of *t*. 2,14,84,492,2870 are 0-*balancing numbers* with 0-*balancers* 1,6,35,204,1189 respectively. 5,34,203,1188,6929 are 1-*balancing numbers* with 1-*balancers* 2,14,84,492,2870 respectively. 3,8,25,54,153 are 2-*balancing numbers* with 2-*balancers* 1,3,10,22,63 respectively. 6,11,45,74,272 are 3-*balancing numbers* with 3-*balancers* 2,4,18,30,112 respectively. 9,14,65,94,391 are 4-*balancing numbers* with 4-*balancers* 3,5,26,38,161 respectively.

Here we note that the 0-*balancing numbers* are the *co-balancing numbers* [8] and the 1- *balancing numbers* are one less than the *balancing numbers* [1].

2. *t*-BALANCING NUMBERS

A positive integer *n* is called a *t-balancing number* if

$$1+2+\cdots\ldots\ldots+n = (n+1+t)+(n+2+t)+\ \ldots\ldots\ldots+(n+r+t) \quad \ldots.. \quad (1)$$

for some positive integer *r*, called the *t-balancer*. From equation (1) we get

$$n=\frac{1}{2}\left((2r-1)+\sqrt{8r^2+8rt+1}\right) \quad \ldots\ldots.. \quad (2)$$

Thus n is a *t-balancing number* if $\sqrt{8r^2+8rt+1}$ is a perfect square. Let $\sqrt{8r^2+8rt+1} = y$
$\Longrightarrow 8r^2+8rt+1 = y^2$, after solving this we get, $2(2r+t)^2 - y^2 = 2t^2-1$.

Put $x = 2r+t$. Since $r>0, t>0$ so $x>0$.
Thus $2x^2 - y^2 = 2t^2 - 1$ …… (3)

So, we need to find integer solution of equation (3). Clearly $x = t$ and $y = 1$ solves equation (3).
Consider the related Pell's equation $y^2 - 2x^2 = 1$ …… (4)
Let the solution of Pell's equation be $\{x_n\}_1^\infty$ and $\{y_n\}_1^\infty$. Consider the following expression,

$$(2t^2-1)(y_n^2 - 2x_n^2) = 2t^2y_n^2 + 2x_n^2 - (4t^2x_n^2 + y_n^2)$$
$$= 2t^2y_n^2 + 2x_n^2 + 2ty_n.2x_n - (4t^2x_n^2 + y_n^2 + 2t\,y_n.2x_n)$$
$$= 2(ty_n + x_n)^2 - (2tx_n + y_n)^2, \text{ which implies,}$$
$$(2t^2-1) = 2(ty_n + x_n)^2 - (2tx_n + y_n)^2 \quad \text{as} \quad y_n^2 - 2x_n^2 = 1.$$

So $X_n = ty_n + x_n$ *and* $Y_n = 2tx_n + y_n$ are the integer solution of equation (3) where x_n and y_n are the solution of the related Pell's equation(4). Solving the Pell's equation (since $x = \pm 2$, $y = \pm 3$ gives us the basic solution of (4)) we have,

$$x_n = \frac{1}{2\sqrt{2}}\{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\}$$

and

$$y_n = \frac{1}{2}\{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\} \quad …….(5)$$

We have taken only $x = 2, y = 3$ for generating this expression. Later we take the remaining values of the associated Pell' equation and try them out for closure's sake.
Substituting x_n, y_n of equation (5) in $X_n = ty_n + x_n$ *and* $Y_n = 2tx_n + y_n$ we get,
$$2\sqrt{2}X_n = \{(3+2\sqrt{2})^n(1+t\sqrt{2}) - (3-2\sqrt{2})^n(1-t\sqrt{2})\}$$
and

$$2Y_n = \{(3+2\sqrt{2})^n(1+t\sqrt{2}) - (3-2\sqrt{2})^n(-1+t\sqrt{2})\} \qquad \text{……… (6)}$$

There are in total of sixteen possible solutions, i.e. four possible values of the base solution of the associated Pell's equation (4) and four possible values of the base solution of the original equation (3). Taking $x = 2, y = 3$ as the base solution of the associated Pell's equation, makes the elements of x_n and y_n positive for all *n*. If we substitute other values like $x = 2$, $y = -3$ or $x = -2$, $y = 3$ or $x = -2$, $y = -3$, then the sequences x_n and y_n have got both negative values as well as positive values.

Now as $X_n = ty_n + x_n$ *and* $Y_n = 2tx_n + y_n$ if we substitute x_n and y_n obtained from the base values $x = 2$, $y = -3$ or $x = -2$, $y = 3$ or $x = -2$, $y = -3$,we can always choose a large enough *t* so that the resulting X_n and Y_n are negative. But the constraints of the problem clearly specify that X_n and Y_n should be positive. Therefore, the only base solution pair that satisfies the constraints are $x = 2$ and $y = 3$. Now x_n and y_n obtained from equation (5) clearly suggest that $x_n < y_n$ for all values of *n*. Therefore, this makes $X_n = ty_n + x_n$ and $X_n = ty_n - x_n$ both positive in sign and the other set of solutions, i.e. $X_n = -ty_n + x_n$ and $X_n = -ty_n - x_n$ both negative in sign. Similar holds for Y_n as well.

So, substituting expressions of x_n and y_n obtained in (5) in these equations, i.e. $X_n = ty_n \pm x_n$ *and* $Y_n = 2tx_n \pm y_n$, we get the following expressions:

$$2\sqrt{2}\,X_n = \{(3+2\sqrt{2})^n(1+t\sqrt{2}) - (3-2\sqrt{2})^n(1-t\sqrt{2})\},$$
$$2Y_n = \{(3+2\sqrt{2})^n(1+t\sqrt{2}) - (3-2\sqrt{2})^n(-1+t\sqrt{2})\}$$

And

$$2\sqrt{2}\,X_n = \{(3-2\sqrt{2})^n(1+t\sqrt{2}) - (3+2\sqrt{2})^n(1-t\sqrt{2})\},$$
$$2Y_n = \{(3+2\sqrt{2})^n(-1+t\sqrt{2}) - (3-2\sqrt{2})^n(1+t\sqrt{2})\} \qquad \text{……..(7)}$$

Now we have got two sequences of X_n and two sequences of Y_n in equation (7).

Using the definition of X_n and Y_n we shall get recurrence relations of X_n and Y_n.

In fact, both the sequences (i.e. two sequences of X_n and two sequences of Y_n) individually satisfy, $Y_n = 6Y_{n-1} - Y_{n-2}$ and $X_n = 6X_{n-1} - X_{n-2}$

Note: Since there are two sequences of X_n(as obtained from expression (7)), each X_n term in the first sequence is surrounded by 2 X_n terms of the second and vice versa. More simply, let $X_{n,1}$ be the generalized *n*th term of the first sequence and $X_{n,2}$ be the generalized *n*th term of the second sequence. Then we have

$$\ldots < X_{n,2} < X_{n,1} < X_{n+1,2} < X_{n+1,1} < \cdots$$

which is simply clear by asserting the first expression in equation (7) to be $X_{n,1}$ and second expression in equation (7) to be $X_{n,2}$. Similar holds for Y_n as well.
So our recurrence relations changes to

$$Y_n = 6Y_{n-2} - Y_{n-4} \text{ and } X_n = 6X_{n-2} - X_{n-4} \qquad \ldots\ldots (8)$$

Also, from (2) and definition of X_n and Y_n, we get that if B_n^t denotes the n^{th} *t-balancing number*, then $B_n^t = \frac{1}{2}(X_n + Y_n - (t+1))$

Putting the values of X_n and Y_n from (8) we get,
$B_n^t = 6\,B_{n-2}^t - B_{n-4}^t + 2(t+1), t \geq 2$
which is the generalized recurrence relation of *t-balancing numbers*.

3. *PROPERTIES OF t-BALANCING NUMBERS*

The original equation
$$1 + 2 + \cdots \ldots \ldots + n = (n+1+t) + (n+2+t) + \ldots \ldots \ldots + (n+r+t)$$
can be reduced to

$$r^2 + (2t + 2n + 1)r - (n^2 + n) = 0 \qquad \ldots\ldots (9)$$

Therefore, *n* is a *t-balancing number* if and only if the discriminant of equation (1) i.e.

$$8n^2 + 8n(1+t) + (2t+1)^2 \qquad \text{....... (10)}$$

is a perfect square.

Consider the function $f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$.

***Theorem* 3.1 :-** *If x is a t-balancing number, then*

$$f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$$

is also a t-balancing number.

Proof :- Let $f(x) = u$ .Then $x < u$ and

$$x = 3u + (t+1) - \sqrt{8u^2 + 8u(1+t) + (2t+1)^2} \qquad \text{.......... (11)}$$

Now since *x* is a *t-balancing number*, $\sqrt{8u^2 + 8u(1+t) + (2t+1)^2}$ must be an integer (because of (11)) or $8u^2 + 8u(1+t) + (2t+1)^2$ must be a perfect square. This and Equation (10) implies that *u* is a *t-balancing number*. But $f(x) = u$ .Therefore $f(x)$ is a *t-balancing number*. ■

***Theorem* 3.2 :-** *If x is the n^{th} t-balancing number, then the $(n+2)^{th}$ t-balancing numberis*

$$f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2},$$

and consequently, the $(n-2)^{th}$ t-balancing number is

$$\tilde{f}(x) = 3x + (t+1) - \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}.$$

Proof :- The function $f : [0, \infty) \longrightarrow [3t+2, \infty)$,defined by

$$f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$$

is strictly increasing since

$$f'(x) = 3 + \frac{4(2x+1+t)}{\sqrt{8x^2+8x(1+t)+(2t+1)^2}} > 0$$

Also f is one-to-one and $x < f(x)$ for all $x \geq 0$. This shows that f^{-1} exists and is also strictly increasing with $f^{-1}(x) < x$. It is easy to prove that

$$f^{-1}(x) = 3x + (t+1) - \sqrt{8x^2+8x(1+t)+(2t+1)^2},$$

Let $f^{-1}(x) = g(x)$ and since

$$8\big(g(x)\big)^2 + 8(1+t)g(x) + (2t+1)^2 = [3\sqrt{8x^2+8x(1+t)+(2t+1)^2} - 8x - 4(t+1)]^2,$$

It follows that $g(x)$ is also a *t-balancing number*. We complete the remaining part of the proof by induction.

The first three *t-balancing numbers* of one of the sequences are $b_1 = 3t+2$, $b_2 = 20t+14$, $b_3 = 119t+84$ and satisfy the relationship $f(b_1) = b_2$ and $f(b_2) = b_3$ which generate the even termed *t-balancing numbers* and the first three *t-balancing numbers* of the other sequence are $b_1 = 3t-3$, $b_2 = 20t-15$, $b_3 = 119t-85$ and satisfy the relationship $f(b_1) = b_2$ and $f(b_2) = b_3$ which generate the odd termed *t-balancing numbers* (the two sets of *t-balancing numbers* namely even termed *t-balancing numbers* and odd termed *t-balancing numbers* is due to the two sequences $\{X_{n,1}\}$ and $\{X_{n,2}\}$ respectively). Let us assume that there is no even (or odd) *t-balancing numbers* between $X_{n-1,1}$ (or $X_{n-1,2}$) and $X_{n,1}$(or $X_{n,2}$) for n = 1,2, …. ,k. To complete the induction, we need to prove that there is no *t-balancing number* between $X_{k,1}$(or $X_{k,2}$) and $X_{k+1,1}$(or $X_{k+1,2}$). Let us assume to the contrary that there exists a *t-balancing number* y between $b_k = X_{k,1}$(or $X_{k,2}$) and $b_{k+1} = X_{k+1,1}$(or $X_{k+1,2}$). Then

$b_k < y < b_{k+1}$

Implies

$$f^{-1}(b_k) < f^{-1}(y) < f^{-1}(b_{k+1})$$

which, in tern, implies

$b_{k-1} < f^{-1}(y) < b_k$

where $b_{k-1} = X_{k-1,1}$ (or $X_{k-1,2}$) and $b_k = X_{k,1}$ (or $X_{k,2}$).
Since *y* is a *t-balancing number*, by the first part of this theorem, $f^{-1}(y)$ is also a *t-balancing number* and the existence of a *t-balancing number* between b_{k-1} and b_k is a contradiction to our assumption that there is no *t-balancing number* between b_{k-1} and b_k. ■

Thus it is evident from the above theorem that, if $B_n^t = x$ is an even (or odd) termed *t-balancing number*, then the next even (or odd) termed *t-balancing number* is $B_{n+2}^t = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$ and the previous even (or odd) termed *t-balancing number* is
$B_{n-2}^t = 3x + (t+1) - \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$.

***Theorem* 3.3** :- $[B_n^t - (t+1)]^2 - B_{n+2}^t B_{n-2}^t = (2t+1)^2$.

Proof :- We know that the *t-balancing numbers* satisfy the recurrence relation,

$$B_{n+2}^t = 6\,B_n^t - B_{n-2}^t + 2(t+1)$$

$$\Rightarrow \frac{B_{n+2}^t + B_{n-2}^t - 2(t+1)}{B_n^t} = 6$$

Similarly from the equation,

$$B_n^t = 6\,B_{n-2}^t - B_{n-4}^t + 2(t+1)$$

we get,

$$\frac{B_n^t + B_{n-4}^t - 2(t+1)}{B_{n-2}^t} = 6$$

Thus

$$\frac{B_{n+2}^t + B_{n-2}^t - 2(t+1)}{B_n^t} = \frac{B_n^t + B_{n-4}^t - 2(t+1)}{B_{n-2}^t}$$

$\Longrightarrow (B_n^t)^2 + B_n^t B_{n-4}^t - 2(t+1) B_n^t = B_{n+2}^t B_{n-2}^t + (B_{n-2}^t)^2 - 2(t+1) B_{n-2}^t$

$\Longrightarrow \left(B_n^t - (t+1)\right)^2 - B_{n+2}^t B_{n-2}^t = \left(B_{n-2}^t - (t+1)\right)^2 - B_n^t B_{n-4}^t.$

On similar lines, we get,

$$\left(B_{n-2}^t - (t+1)\right)^2 - B_n^t B_{n-4}^t = \left(B_{n-4}^t - (t+1)\right)^2 - B_{n-2}^t B_{n-6}^t.$$

Thus, we have,

$$\left(B_n^t - (t+1)\right)^2 - B_{n+2}^t B_{n-2}^t = \left(B_{n-4}^t - (t+1)\right)^2 - B_{n-2}^t B_{n-6}^t.$$

Continuing in this fashion, we have,

$$\begin{aligned}\left(B_n^t - (t+1)\right)^2 - B_{n+2}^t B_{n-2}^t &= \left(B_3^t - (t+1)\right)^2 - B_5^t B_1^t, \quad \textit{if n is odd.}\\ &= \left(B_4^t - (t+1)\right)^2 - B_6^t B_2^t, \quad \textit{if n is even.}\end{aligned}$$

Again, we have,

$B_1^t = 3t - 3;\ B_3^t = 20t - 15;\ B_5^t = 119t - 85.$ Also, $B_2^t = 3t + 2;\ B_4^t = 20t + 14;\ B_6^t = 119t + 84.$

(These values are obtained from X_n and Y_n which are the variables in the original equation (3) of section 2, along with the definition of *t-balancing number* as mentioned in section 2, i.e. $B_n^t = \frac{1}{2}(X_n + Y_n - (t+1))$.

So putting these values, for both the cases (*i.e. n is odd or even*) we get that,
$[B_n^t - (t+1)]^2 - B_{n+2}^t B_{n-2}^t = (2t+1)^2.$ ■

Theorem* 3.4 *:- *The Lucas t-balancing number* c_n^t **,** *satisfy the relation* $c_{n+2}^t = 6\, c_n^t - c_{n-2}^t$.

Proof :- We define *Lucas t-balancing numbers* as:

$$c_n^t = \sqrt{8(B_n^t)^2 + 8(1+t)\, B_n^t + (2t+1)^2}$$

where B_n^t is the n^{th} *t-balancing number*. Therefore, we have

$$(c_{n+2}^t)^2 = 8(B_{n+2}^t)^2 + 8(1+t)\, B_{n+2}^t + (2t+1)^2$$

Substitute $B_{n+2}^t = 3\, B_n^t + (t+1) + \sqrt{8\,(B_n^t)^2 + 8(1+t)\, B_n^t + (2t+1)^2}$ into the above equation, we get $(c_{n+2}^t)^2 = [3c_n^t + 8\, B_n^t + 4(t+1)]^2$
which implies $c_{n+2}^t = 3c_n^t + 8\, B_n^t + 4(t+1)$ ……… (12)

On similar lines, we can prove,

$$(c_{n-2}^t)^2 = 8(B_{n-2}^t)^2 + 8(1+t)\, B_{n-2}^t + (2t+1)^2$$

Substitute $B_{n-2}^t = 3\, B_n^t + (t+1) - \sqrt{8\,(B_n^t)^2 + 8(1+t)\, B_n^t + (2t+1)^2}$ into the above equation, we get

$$(c_{n-2}^t)^2 = [3c_n^t - 8\, B_n^t - 4(t+1)]^2$$

which implies

$$c_{n-2}^t = 3c_n^t - 8\, B_n^t - 4(t+1) \quad \text{………..(13)}$$

So adding (12) and (13), we get,

$$c_{n+2}^t + c_{n-2}^t = 6\, c_n^t$$

which implies, $c_{n+2}^t = 6\, c_n^t - c_{n-2}^t$. ■

The above recurrence relation is a homogenous recurrence relation of fourth order. Putting $c_n^t = z^n$, in the equation, we get,

$$z^4 - 6z^2 + 1 = 0$$

Putting $z^2 = y$, we get,

$$y^2 - 6y + 1 = 0$$

which gives,

$y = 3 \pm 2\sqrt{2} \implies z = \sqrt{3 \pm 2\sqrt{2}}$

Hence $c_n^t = A\,\alpha^n + B\beta^n$, where $\alpha = \sqrt{3 + 2\sqrt{2}}$ and $\beta = \sqrt{3 - 2\sqrt{2}}$.

Thus we get the following result:

Theorem 3.5 :- $c_{n+2}^t = 6\,c_n^t - c_{n-2}^t$, *where* $c_n^t = A\,\alpha^n + B\beta^n$, *with* $\alpha = \sqrt{3 + 2\sqrt{2}}$ *and* $\beta = \sqrt{3 - 2\sqrt{2}}$.

We have found out recurrence relation between *t-balancing numbers*. Let us find out relation between the *t-balancers*. The equation,

$$1 + 2 + \cdots \ldots \ldots + n = (n + 1 + t) + (n + 2 + t) + \ldots \ldots \ldots + (n + r + t)$$

can be modified to give

$$r^2 + (2t + 2n + 1)r - (n^2 + n) = 0$$

$$\implies r = \frac{1}{2}[-(2n + 2t + 1) + \sqrt{8n^2 + 8n(1 + t) + (2t + 1)^2}] \quad \ldots\ldots\ldots(14)$$

More specifically, let R_n^t and B_n^t denote the n^{th} *t-balancer* and n^{th} *t-balancing number* respectively.

So from equation (6),we have, R_{n-2}^t

$$= \frac{1}{2}[-(2\,B_{n-2}^t + 2t + 1) + \sqrt{8\,(B_{n-2}^t)^2 + 8(1 + t)\,B_{n-2}^t + (2t + 1)^2}] \quad \ldots (15)$$

And R_{n+2}^t
$= \frac{1}{2}[-(2\,B_{n+2}^t + 2t + 1) + \sqrt{8\,(B_{n+2}^t)^2 + 8(1+t)\,B_{n+2}^t + (2t+1)^2}]$ …(16)

But from result (12) of Theorem 3.4,after applying the definition of Lucas *t-balancing numbers* we have,

$$8\,(B_{n+2}^t)^2 + 8(1+t)\,B_{n+2}^t + (2t+1)^2$$
$$= [3\sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2} + 8\,B_n^t + 4(t+1)]^2.$$

So after substituting B_{n+2}^t in terms of B_n^t , equation (16) reduces to
$2R_{n+2}^t = 2\,B_n^t + 1 + \sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2}$ ….. (17)

On similar lines, result (13) of Theorem 3.4, after applying the definition of Lucas *t-balancing numbers* we have,

$$8\,(B_{n-2}^t)^2 + 8(1+t)\,B_{n-2}^t + (2t+1)^2$$
$$= [3\sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2} - 8\,B_n^t - 4(t+1)]^2$$

So after substituting B_{n-2}^t in terms of B_n^t , equation (15) reduces to

$2R_{n-2}^t = -8t - 7 - 14\,B_n^t + 5\sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2}$ …… (18)

Adding (17) and (18), we get,

$$2(R_{n+2}^t + R_{n-2}^t)$$
$$= 6[-(2t + 1 + 2\,B_n^t) + \sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2}] + 4t$$

But by (14),
$[-(2t + 1 + 2\,B_n^t) + \sqrt{8\,(B_n^t)^2 + 8(1+t)\,B_n^t + (2t+1)^2}] = 2R_n^t$

Thus we have,

$$R_{n+2}^t + R_{n-2}^t = 6\,R_n^t + 2t \implies R_{n+2}^t = 6\,R_n^t - R_{n-2}^t + 2t\ .$$

Thus we have proved,

***Theorem* 3.6 :-** *The recurrence relation for t-balancer is,*
$R_{n+2}^{t} = 6\, R_{n}^{t} - R_{n-2}^{t} + 2t$.

Let us find out the Diophantine equation that *t-balancing numbers* solve. Start from equation

$$1 + 2 + \dots\dots\dots + b = (b + t + 1) + (b + t + 2) \dots\dots\dots + (b + t + r),$$

where *b* is a *t-balancing number*. Let $u = b + t + r,$ we have,

$$\frac{b(b+1)}{2} = \frac{u(u+1)}{2} - \frac{(b+t)(b+t+1)}{2},$$

On solving this a little, we get,

$$b = \frac{1}{2}\left(-(1+t) + \sqrt{2u^2 + 2u + 1 - t^2}\,\right)$$

Since *b* is a *t-balancing number*, $2u^2 + 2u + 1 - t^2$ is a perfect square.
Also,

$$2u^2 + 2u + 1 - t^2 = u^2 + (u+1)^2 - t^2$$

Thus we can say that the Diophantine equation, $x^2 + (x+1)^2 - y^2 = z^2$,
is solved by
$x = u,\ y = t\ \ and\ z = \sqrt{2u^2 + 2u + 1 - t^2}$.

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